

A SHORTER SCHOOL GEOMETRY

Books by W. J. WALKER, M.A.

A NEW SCHOOL ALGEBRA

With or without Answers.

Also in Parts. Part I (including quadratics in one unknown).

Parts I and II (including Progressions, Indices and Surds).

Parts II and III (to the Binomial and Exponential Theorems).

A NEW SCHOOL GEOMETRY

Also in two parts.

Also issued separately. Stages I and II.

A SHORTER SCHOOL GEOMETRY

A simplified form of New School Geometry.

A SHORTER SCHOOL GEOMETRY

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PREFACE

THIS book is a simplified form of the author's "New School Geometry," at present (1939) in its twelfth edition. Since the original book was published, in 1927, the teaching of Geometry has, in many schools, tended to become less and less complex. One reason for this fact may be that more children of average intelligence are being prepared for the various School Certificate Examinations; another reason certainly is that under the increasing pressure of other subjects less time can be devoted to the older subjects, such as Geometry, than used to be the case. It was therefore felt that many teachers would welcome the opportunity of having a text-book which contained the full bookwork of the ordinary Certificate course and also plenty of riders simple enough for the average pupil, but which omitted the more difficult applications of the subject. At the same time all the really essential things will, it is hoped, be found to be included in this simplified edition, as also are the main characteristics of the original book. For instance, the references in the Propositions are set out in phrases—not in numbered references to earlier Theorems—an important feature which the original book was the first to introduce and use consistently.

The original numbering, both of the Figures and of the sets of Examples, has been retained, so that the two forms of the book can be used side by side when deemed necessary.

The author hopes that teachers will welcome this simpler form of a School Geometry, which presents the subject in a way that is mathematically rigid and useful practically, both by the care with which the bookwork is set out and by reason of the large store of simple riders of the various types necessary, numerical and practical as well as theoretical.

An addition has been made in this reprint (1948) consisting of sets of Revision Examples which deal with the various parts of the work. They have been made fairly easy in order that they may be used either for homework or for revision.

W. J. WALKER.

CONTENTS

STAGE I—ELEMENTARY IDEAS

	PAGE
Solids, Surfaces, Lines, Points	1
Measuring and Drawing Straight Lines	4
Angles and their Measurement	7
Use of Instruments	16
Drawing to Scale	18
Geometrical Solids	21

STAGE II—FUNDAMENTAL TRUTHS

Angles at a Point—Adjacent and Vertically Opposite	24
Directions of Lines—Parallel Lines	29
Angles of a Triangle	37
Angles of a Quadrilateral and Polygon	39
Copying a Triangle—Congruences	39
Drawing of Triangles, etc.	46
The sides of a Right-angled Triangle	48
Symmetrical Figures	49
Similar Figures	51

STAGE III—THEORETICAL GEOMETRY

SECTION I—ANGLES, PARALLELS, AND CONGRUENCES

Meaning of Terms	55
Easy Arguments	57
Angles at a Point	58
Parallels	61
Angles of a Triangle	66

CONTENTS

vii

	PAGE
Angles of a Polygon	68
"Two Sides and Included Angle" Congruence	70
"Two Angles and a Side" Congruence	71
Isosceles Triangle	74
"Three Sides" Congruence	77
"Right Angle, Hypotenuse, and one Side" Congruence	78
Bisection of an Angle and of a Line	81
A Line Perpendicular to a Line	83
Constructions without Protractor or Set-square	87
Inequalities	88
The Parallelogram—its Angles, Sides, and Diagonals	92
Equal Intercepts in a Triangle and in Three Parallels	102
Equal Division of a Line	106

SECTION II—AREAS

Area of a Rectangle	110
Areas of Parallelogram and Triangle	112
Areas of Trapezium and Quadrilateral	118
A Triangle equal to a Quadrilateral	121
Theorem of Pythagoras	125

SECTION III—LOCI

Elementary Ideas of a Locus	132
Practical Exercises on Loci	133
Theorems on Loci	137
Intersection of Loci	139

SECTION IV—THE CIRCLE

Chords of a Circle	142
Circle through Three Points	144
Equal Chords of a Circle	148
Angles at the Centre and Circumference of a Circle	150
Cyclic Quadrilateral	155

	PAGE
Concyclic Points	162
Equal Angles, Arcs and Chords	164
Tangent to a Circle	168
Inscribed and Escribed Circles of a Triangle	176
Circles in Contact	178
Construction of Circles	181
Angles in the Alternate Segment	185
Common Tangents in Two Circles	188
Inscribed and Circumscribed Regular Polygons	194
Circumference and Area of a Circle	197, 198
Arcs and Sectors	199

SECTION V—RECTANGLES AND SQUARES

Figures corresponding to Algebraical Identities	203
Projection on a Line	208
Extensions of Pythagoras	210
Theorem of Apollonius	212
Rectangle Contained by Segments of a Chord	215
Secant and Tangent to a Circle	216
A Square equal to a Rectangle	221

SECTION VI—SIMILAR FIGURES

Division of a Line in a given Ratio	224
A Line Parallel to a Side of a Triangle	228
Fourth and Third Proportionals	231
Bisector of an Angle of a Triangle	234
The Circle of Apollonius	236
Similar Figures	238
Similar Triangles	239
Proof of Pythagoras by Similar Triangles	246
Mean Proportional	246
Areas of Similar Triangles	250
A Mental Test	253
Revision Examples	257

STAGE I.

ELEMENTARY IDEAS.

ELEMENTARY IDEAS.

Solids in Geometry.

A Solid is the name we give to any body with a *definite boundary*. It need not be of a really solid nature; for example, a carpet and a piece of string are both called solids; so also is an india-rubber ball, though this is, in one sense, hollow.

The shape of a solid may be quite irregular, as in the case of a boot, a kettle, or a cricket-shirt. On the other hand there are a few solids with actual geometrical names. There are, for instance, the cube, the cone, the pyramid, and others, which we should examine and learn by name.

Let us learn the nature of all solids by considering a brick; the shape of this is often called a rectangular block. Notice that it has *length, and breadth, and thickness*, three dimensions, as they are called. Every solid that we can think of has the same three dimensions, though it may not be as easy to measure them as in the case of a brick.

Surfaces.

A solid is always bounded by Surfaces. A surface is merely the boundary of a solid. It has no thickness at all, so that it has two dimensions only, *length and breadth*. Distinguish between a page of this book and the upper surface of the page. The page itself is a solid, though its thickness happens to be very small, and the solid page is,

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of course, made of paper. The top surface of the page has no thickness, nor is it made of paper at all; for if it were, it would need to have some thickness. In fact the top surface is merely where the paper ends and the air above it begins.

Surfaces may be either *plane* or *curved*. The surface of a cup is curved; the surfaces of the walls of a room, if well made, are plane. If it is desired to find out whether a surface is plane or not, an ordinary ruler should be taken and laid across it in many positions. If the surface is plane, the ruler will *always* lie flat against it.

A *plane* surface is often spoken of as flat, but it is better to use the word "plane," as a flat surface might be taken to mean one that is horizontal, like the surface of water at rest. The wall of a room may be plane, but we should hardly call it flat.

Lines.

Two surfaces meet in a line. A line, therefore, has neither breadth nor thickness; it has one dimension only, *length*. Think of the line where the floor of the room meets one of the walls. It is not made of the plaster which forms the walls, or of the wood which forms the floor, or of the air in the room. It is simply where the air, the wood and the plaster meet.

Lines can be either *straight* or *curved*. (We use the word curved to include all sorts of irregularities.) If a curved piece of string connects two nails in a wall, it is evident that the string would have to be pulled at one end tighter and tighter before it would be a straight line joining the nails. Thus a *straight line* gives us the *shortest distance between its two end points*. A straight line is *even* from end to end. The edge of a ruler is, or should be, straight, and so we use a good ruler and a well-sharpened pencil to draw straight lines. If you want to find out whether a ruler edge is straight or not, take a pencil and use the ruler to draw a line AB on your paper. Reverse the ruler

and use the same edge reversed to draw again a line from A to B. If the two lines AB are the same, and have no space between them, the ruler must have a straight edge.

Points.

Two lines meet in a point. A point, therefore, has neither length nor breadth nor thickness; it has no dimensions at all, no size, though it has position. We can think of the right-hand top corner of this page as being a point, where the right-hand edge of the paper meets the top edge.

In drawing a point, make two fine lines cutting one another in a cross ($\times$), and not by a dot of pencil or ink ($\cdot$).

Points, Lines, Surfaces, Solids.

We may begin with a point, instead of with a solid. If a point were to move, it would trace out a line, straight or curved. If a line were to move, it would generally trace out a surface. If a surface were to move, it would generally trace out a solid.

EXAMPLES 1.

1. Give instances of solids, surfaces, lines and points in your classroom.

2. Point out surfaces about you which are (i) plane; (ii) not plane.

3. Point out lines about you which are (i) straight; (ii) not straight.

4. Mention articles that are bought and sold (i) by volume; (ii) by area; (iii) by length.

5. How many (i) surfaces, (ii) edges, (iii) corners has an ordinary brick?

6. How many measurements must I give in order to have a wall built?

7. How many measurements must I give in order to have a wall papered on one side?

8. Is this page a surface or a solid?

9. Move your pencil to show how a straight line can move to trace out (i) a plane surface; (ii) a curved surface; (iii) no surface at all.

10. Is one thread of a spider's web a line, a surface or a solid?

11. Is the best straight line you can draw on your paper really a line? Is it really straight? What sort of a thing is it actually?

12. Why did the Romans make their roads straight? Are there any disadvantages in roads that are straight?

13. Is a coat of varnish on a door a solid or a surface?

14. Name some objects that are bounded partly by curved surfaces and partly by plane surfaces.

QUESTIONS ON THE GEOMETRICAL SOLIDS (see page 21).

15. Can a straight line be made to move so as to describe the curved surface of (i) a cylinder, (ii) a sphere, (iii) a cone?

16. Give the number of faces, edges and corners in each of the following: a cube; a rectangular block; a triangular prism; a cylinder; a cone; a triangular pyramid.

17. Name geometrical solids that are bounded by:

- (i) Plane surfaces only;
- (ii) Curved surfaces only;
- (iii) Plane and curved surfaces.

MEASURING STRAIGHT LINES.

The ruler you use in Geometry should have one edge divided into *inches and tenths of an inch*, and the other edge divided into *centimetres (cm.) and millimetres (mm.)*. Remember that $1 \text{ mm.} = \frac{1}{10} \text{ cm.}$

In measuring, always use decimal, not vulgar, fractions.

In measuring, do not use the "zero" end if it is at all worn; begin, say, at the 1-inch mark instead, subtracting the 1 inch from the measurement obtained. A better result still will be obtained if the ruler is tilted, so as to bring the graduations on the edge of the ruler as close as possible to the line which is being measured.

DRAWING STRAIGHT LINES, ETC.

In all drawing exercises the pupil should use a *hard pencil* (H for preference). Fairly good work can be done with a pencil sharpened to a *fine point* in the ordinary

way, but much better work is possible if the pencil is sharpened to a *chisel edge*, a piece of glass-paper being used afterwards to make the edge as sharp and straight as possible. It is a good thing to hold the pencil against the ruler in a position that is upright or nearly so.

ON DRAWING AND MEASURING STRAIGHT LINES.

A. In inches and tenths; in cm. and mm.

EXAMPLES 2.

1. Draw straight lines of the following lengths, and mark the length over each: (i) 1·7 in.; (ii) 2·6 in.; (iii) 3·7 in.; (iv) 5 in.

2. Guess the length in inches of the two lines AE, BC in Fig. 1; then measure each line and find your errors.

3. Draw, without measuring, a line across the page and one down the page, each 3" long; then measure the lines and find your errors.

4. Measure the lengths AB, BC, CD, DE in inches; check by adding your results and by measuring the whole line.

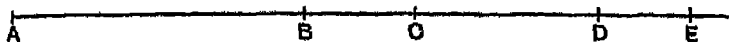


FIG. 1.

5. Draw a line AB across your page; cut off from it a part AC 4·4 cm. long, and then a part CD 2·7 cm. long. How long should AD be? Check by measurement.

6. Measure the lengths PQ, QR, RS in Fig. 2 in centimetres and millimetres; check by calculating and measuring the length of PS.

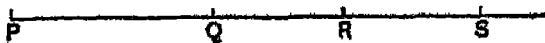


FIG. 2.

7. How many times is the line AE in Fig. 1 contained in the height of this page? Guess an answer and check it by measuring in millimetres and dividing.

B. In inches or centimetres (to two decimal places).

When a line is not an exact number of inches and tenths, the length of the small amount left over has to be guessed. Divide a tenth of an inch into ten equal parts in your mind; each small part is now one-hundredth of an inch, i.e. 0·01 inch. You have now to guess the number of these small parts in the piece of the line still to be measured. If it is a little less than half of $\frac{1}{10}$ inch,

it is taken as $\cdot 03$ in. or $\cdot 04$ in.; if a little less than a full tenth, as $\cdot 08$ in. or $\cdot 09$ in., and so on.

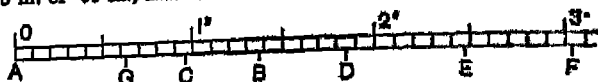


FIG. 3.

Thus in Fig. 3, AB is about 1.87 in., and AG about $\cdot 53$ in.

EXAMPLES 3.

1. Measure in inches to two decimal places the lengths of AC, AD, AE, AF in Fig. 3.



FIG. 4.

2. In Fig. 4 AX is about 2.06 cm. Measure AY, AZ, AW, AH, AJ, AL in cm., mm., and tenths of a mm.

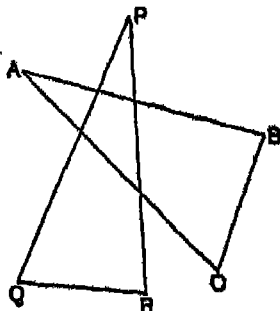


FIG. 5.

3. In Fig. 5 measure AB, BC, CA, PQ, QR, RP in inches and in centimetres; put down the results in tabular form:

Line.	AB	BC	CA	
Length in inches. .				
Length in cm. . .				

4. Draw a line 10 cm. long, measure it in inches, and so find the number of inches in 1 cm.

5. Draw a line 4 inches long, measure it in cm., and so find, in centimetres, how much longer a metre is than a yard.

6. Draw a line 7 cm. long and another 2 inches long. Find, in cm., how much one line is longer than the other.

7. Measure in millimetres the length of three different pages of this book (close to the right-hand edge), and find the average of the three results. (Ans. correct to .1 mm.)

8. Work out the same question in inches (correct to .01 inch).

9. Draw a straight line across the width of your paper and divide it into three equal parts by guessing. Measure the three parts. How much longer is the biggest part than the smallest?

(Work in inches, and then measure again in cm.)

ANGLES.

An Angle is formed by the meeting of two straight lines. The size of the angle depends on *how widely open* the lines (or arms of the angle) are. *The length of these arms does not affect the size of the angle.* For example, the angles x and y are equal; also the angle y is smaller than the angle z , because the arms in y are less open than they are in z .

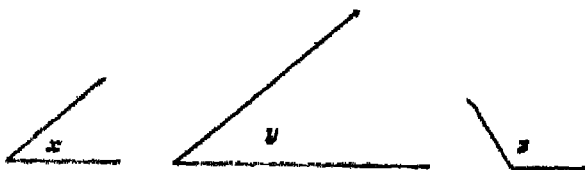


FIG. 6.

An angle may be considered as being made by the *turning* (or rotation) of a line from the position of one arm, say OA, into the position of the other arm, OB; the *size of the angle is the amount of turning* involved.

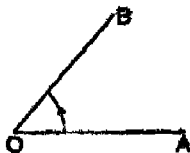
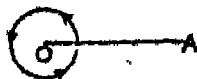


FIG. 7.



A complete turn.

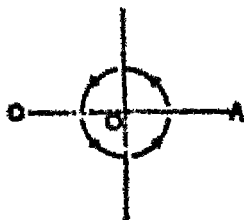
FIG. 8.

If a line rotates from the position OA completely round until it reaches the same position again, it makes, of course, a complete revolution. When it has gone half-way round, the angle described is called a straight angle; when it has gone one quarter of the way round, the angle is called a right angle.



2 straight angles = 1 complete turn.

FIG. 9.



4 right angles = 1 complete turn.

FIG. 10.

An angle less than a right angle is called an acute angle.

An angle greater than one right angle, but less than two right angles, is called an obtuse angle.

An angle greater than two right angles, but less than four right angles, is called a reflex angle.



Acute angle.



Right angle.



Obtuse angle.



Straight angle.



Reflex angle.

FIG. 11.

Angles can be referred to in two ways:

- (i) By one letter (such as x in Fig. 6, and O in Fig. 7) when there is no doubt as to which is the angle referred to;
- (ii) By three letters in all other cases (such as AOB or AOO in Fig. 12).

Notice that the point where the angle is appears in the middle of the three letters used; also that the angle AOB could be called the angle BOA (generally written $\angle AOB$ or $\angle BOA$).

One cannot speak of the angle O in Fig. 12, because there are several angles at O, and it would not be certain which angle was meant. In this book one letter is used when there is no ambiguity, but three letters are used in other cases.

EXAMPLES 4.

1. Mention in Fig. 12 the names of
 - (i) several acute angles;
 - (ii) a right angle;
 - (iii) several obtuse angles.
2. Draw an acute angle AOB, then a right angle BOC, and then an obtuse angle COD, all formed at the same point O.
3. State the size of the angle (in right angles and fractions) which I turn through if I

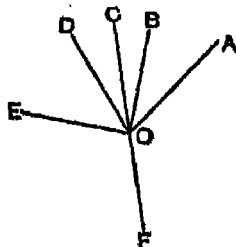


FIG. 12.

- (i) Face North, then turn to the West.
 - (ii) Face East, then turn to the South-East.
 - (iii) Face North-East, then turn and face South-West (a) by turning to the right; (b) by turning to the left.
 - (iv) Face North, then turn to the right until I face West.
(The Mariner's Compass is explained on p. 14.)
 4. State what kinds of angles are to be found in the letters E, A, M.
 - * 5. Find the size of the angle made
 - (i) By the hour-hand of a clock between 1 p.m. and 4.30 p.m.
 - (ii) By the minute-hand between 5 p.m. and 5.25 p.m.
 - (iii) By each hand between 4 p.m. and 4.30 p.m.
 - (iv) By the hands of a clock at
 - (a) 2 o'clock; (b) 3.12 o'clock; (c) 5.30 o'clock;
 - (d) 7.30 " ; (e) 2.24 " ; (f) 10.38 " .
 6. Show, by drawing, that the sum of two acute angles may be either an acute angle, or a right angle, or an obtuse angle.
 7. Sketch angles of the following sizes:
 - $\frac{1}{2}$ right angle; $1\frac{1}{2}$ right angles; 1 right angle; 3 right angles; $3\frac{1}{4}$ right angles.
 8. Fold an irregular piece of paper so that, when it is opened out, the creases form four right angles.
- * Question 5 can be answered in degrees if preferred and taken with Examples 6.

HORIZONTAL AND VERTICAL.

The meanings of these words must be clearly understood. They are used to refer to straight lines and to planes.

Lines.

Any line along the floor (or the ceiling) of a room is horizontal; any *upright* line, such as a string hanging with a lump of lead at the lower end, is vertical. A spirit-level is used to test whether a line is horizontal, and a plumb-line to test whether a line is vertical. Both are used constantly when a house is being built.

Planes.

The floor (or ceiling) of a room is a horizontal plane; each wall is a vertical plane. Notice that *all* lines in a horizontal plane are horizontal; the lines in a vertical plane, however, may be vertical or horizontal or oblique (*i.e.*, neither horizontal nor vertical).

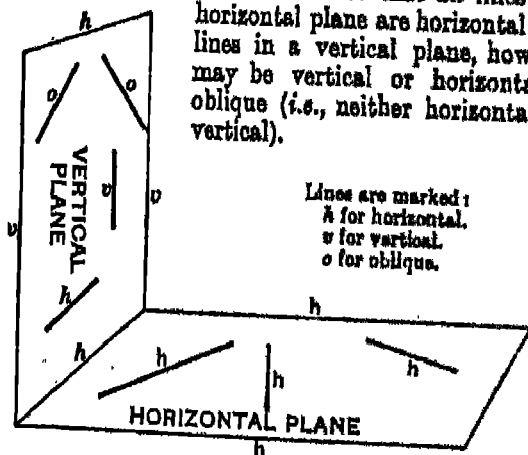


FIG. 18.

Perpendicular. When two lines form a right angle each is said to be perpendicular to the other. Thus any vertical line is perpendicular to any horizontal line that meets it.

EXAMPLES 5.

1. Name several vertical lines in your class-room.
2. Name several horizontal lines in your room.
3. Point horizontally; in how many horizontal directions can you point?
4. Point vertically; in how many vertical directions can you point without moving from your seat?
5. How many vertical lines are there in the room where walls and floor or ceiling meet? How many of these lines are horizontal?
6. How many horizontal planes are there in the room? How many vertical planes?
7. How does a builder test
 - (i) whether the floor is really horizontal?
 - (ii) whether an outside wall of a house is really vertical?
8. If a plane contains a horizontal line, must the plane be horizontal?
9. If a plane contains a vertical line, must the plane be vertical?
10. In what sort of lines will the following kinds of planes meet?
 - (i) a vertical and a vertical.
 - (ii) a vertical and a horizontal.
 - (iii) a horizontal and an oblique.
 - (iv) a vertical and an oblique.
11. Can the following pairs of lines be perpendicular? Are they certain to be perpendicular?

(i) two vertical lines.	(iv) a horizontal and a vertical.
(ii) two horizontal lines.	(v) a horizontal and an oblique.
(iii) two oblique lines.	(vi) a vertical and an oblique.

MEASUREMENT OF ANGLES IN DEGREES.

It is awkward to have to express small angles always in terms of a right angle, and so a right angle is divided into 90 equal parts called degrees.

- 1 right angle = 90 degrees i.e. 90° ;
 1 straight angle = 180 degrees i.e. 180° ;
 1 complete turn = 360 degrees i.e. 360° .

Any angle can be measured in degrees. The instrument used for doing so is called a protractor. Fig. 14 shows

a protractor set to indicate an angle AOX which is equal to 48° ; it also shows an angle BOX which is equal to 132° . The two sets of numbers marked on the protractor run opposite ways, each from 0° to 180° , and they must never be confused. If you ask yourself, before measuring an angle, whether it is acute or obtuse, you will know which numbering is the correct one in each case. A protractor can also be used for *drawing* an angle of any given size, as long as it is less than two right angles (or 180°).

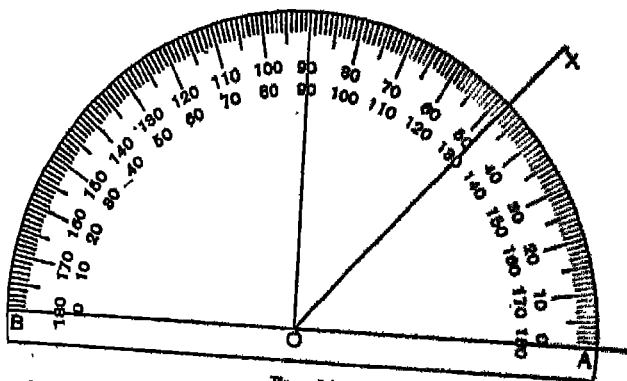


FIG. 14.

EXAMPLES 6.

- Express in degrees:
 $\frac{1}{2}$ right angle, $1\frac{1}{2}$ right angles, $\frac{3}{4}$ right angle, $\frac{1}{4}$ right angle.
- Express as fractions of a right angle:
 45° , 60° , $22\frac{1}{2}^\circ$, 10° , 120° , 135° , 105° , $67\frac{1}{2}^\circ$.
- How many degrees are there in $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{8}$, $\frac{1}{10}$ of a complete turn?
- Measure the three angles of each of your set squares. How many degrees are they intended to have?
- Use your protractor to draw angles of:
 (i) 30° ; (ii) 45° ; (iii) 75° ; (iv) 110° ; (v) 132° ; (vi) 117° .
 (Insert the number of degrees inside each angle.)

6. Measure the angles AOB and BOC in Fig. 15, and, after finding their sum in degrees, check your result by measuring the angle AOC, which is their sum.

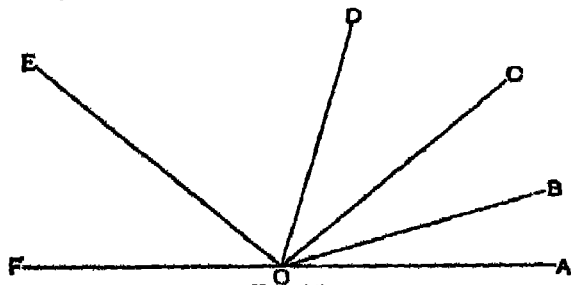


FIG. 15.

7. In the above figure, measure the following angles, and check by addition and by measuring the last angle given.

- (i) AOC, COD; their sum = AOD.
- (ii) AOB, BOE; their sum = AOE.
- (iii) BOD, DOF; their sum = BOF.
- (iv) COD, DOE, EOF; their sum = COF.
- (v) AOD, DOF; their sum is the straight angle AOF.
- (vi) AOC, COE, EOF; their sum is the straight angle AOF.

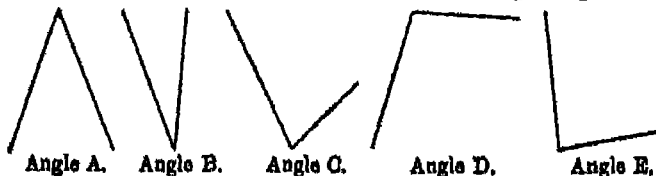


FIG. 16.

8. By inspection only, arrange in order of size the angles A, B, C, D, E, putting the smallest first. (See Fig. 16.)

9. Guess the number of degrees in each of the angles P, Q, R, S, T. Measure each and find the error in each case.

10. Draw angles of the following sizes by guesswork, using your ruler, but not your protractor. Then measure each and find the error in each case.

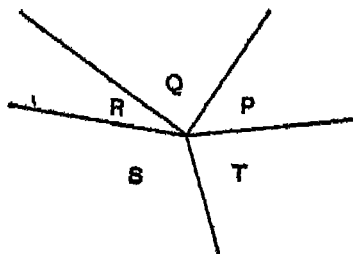


FIG. 17.

35°; 140°; 65°; 115°; 50°.

NEW SCHOOL GEOMETRY

THE MARINER'S COMPASS.

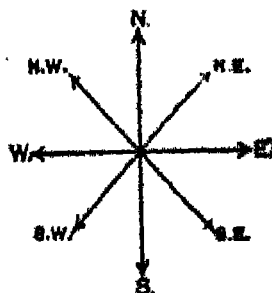


FIG. 18.

The direction in which one point lies, when seen from another, is often given by the reading of a Mariner's Compass. The Compass card has four principal directions, N., S., E., W. Bisecting the angles between these are the directions N.E., N.W., S.E., S.W. Thus we have eight directions separated by angles of 45° . The sailor obtains smaller angles by bisecting these twice over, thus obtaining angles of $\frac{1}{2}$ of 45° or $11\frac{1}{2}^\circ$. We can, however, express any direction in a more simple manner. For instance the first direction shown in Fig. 19 can be described as N. 25° W. (or 25° W. of N.).

The same direction is given by W. 65° N. (or 65° N. of W.). So also E. 25° N. or 25° N. of E. means 25° on the North side of East.

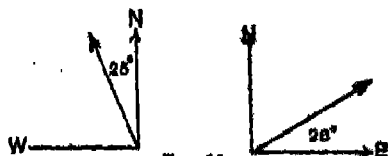


FIG. 19.

EXAMPLES 7.

(In these exercises assume that the Northerly direction is towards the top of the page.)

1. Draw figures illustrating the following directions, and underneath each put the direction:

(i) N. 30° E., (ii) N. 50° W.,
(iii) E. 20° S., (iv) W. 48° N.,
(v) S. 42° W., (vi) 40° W. of N.,
(vii) 72° N. of E., (viii) 31° W.
of S.

2. Give the directions of the lines in Fig. 20 as seen from the central point.

3. Give three other names for each of the following directions, and illustrate with a freehand figure:

W. 47° N., S. 25° E., N. 37° E.,
E. 27° S., E. 70° N.

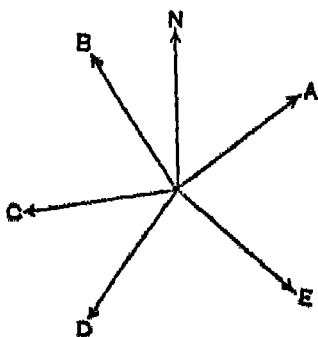


FIG. 20.

THE CIRCLE.

Fig. 21 shows a circle, the space enclosed being the circle, and the curved boundary being called the circumference.

Every point on the circumference is at the same distance from the fixed point O.

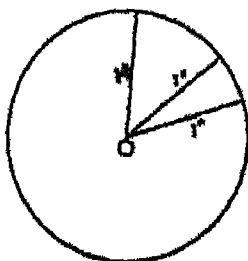


FIG. 21.

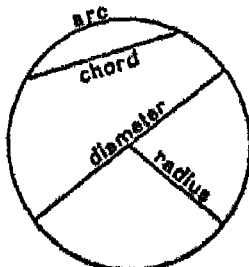


FIG. 22.

O is called the centre of the circle; each line from O to the circumference is called a radius. Thus:

A circle is a plane figure bounded by a curved line, every point on which is at the same distance from a certain fixed point inside the figure. This point is called the centre of the circle, and every line from the centre to any point on the boundary is called a radius. The curved boundary of the circle is called the circumference. Any portion of the circumference is called an arc. (See Fig. 22.)

A chord of a circle is a straight line joining any two points on the circumference. (See Fig. 22.)

A diameter of a circle is a chord which passes through the centre.

Hence each diameter of a circle is twice the radius.
The plural of radius is radii.

USE OF INSTRUMENTS.

We have already shown how a protractor can be used both to measure angles and to draw angles of a given size.

It can be used when we are dealing with a right angle, but as a rule a set-square is used to make right angles and perpendiculars as well as parallels.

1. To draw a line perpendicular to a given line AB at a given point O in the line.

Place the set-square so that one of the shorter edges is along the line AB and the right angle corner is at the given point O .

Use your pencil to draw a line along PO on your paper.

Then since $\angle POQ$ is a right angle, PO is perpendicular to AB and is drawn through the given point O .



FIG. 23.

If the set-square is worn at the corner O , it cannot be used in this way to give an accurate result; it would be better in this case to use a protractor in order to make the angles at O right angles.

2. To draw a line perpendicular to a given line AB from a given point O outside the line.

This is similar to the last construction. The set-square must be placed so that one of the shorter arms is along AB and the other goes through O . Draw the line PR .

Then PR is the perpendicular required.



FIG. 24.

3. To draw a line parallel to a given line AB through a given point O .

Place your set-square with its longest edge PR along the line AB , and then put a ruler exactly along PQ , one of the other edges.

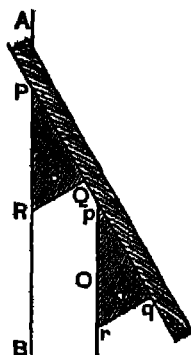


FIG. 25.

Keep the ruler firmly in place and slide the set-square until the longest edge just passes through O (in the position pr).

Then pr is a line through O parallel to AB .

While the set-square is being moved PQ keeps on occupying different parts of the straight line Pq , and the direction of PR also remains the same during the sliding, so that pr and PR are in the same direction, that is, they are parallel.

In practice the pupil must be careful to have his pencil well sharpened either with an ordinary point or with a chisel point. Even then it will not be easy to draw a line along an edge of the set-square to go through a particular point O , if the set-square is placed to cover O exactly. It is better to place the set-square edge to go nearly through O , so that when the pencil is used the line is drawn to go exactly through O . It is easily seen how much allowance must be made in practice for the fact that the pencil does not draw a line exactly along the edge of the set-square.

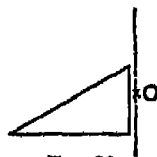


FIG. 26.

DRAWING TO SCALE.

In drawing to scale a figure that is bounded by straight lines (such as the playground shown in Fig. 27), it can be seen that:

(i) The lengths of all straight lines are increased (or reduced) in the same ratio. Thus, if a plan, as it is called, of the playground is made on a scale of 1 inch to 1 yard, every straight piece of wall is represented by a straight line $\frac{1}{3}$ as long as the original (1 inch = $\frac{1}{3}$ of a yard). This fraction $\frac{1}{3}$ is called the Representative Fraction (or R.F.) of the scale. If the scale used is 1 foot to the mile, the R.F. is $\frac{1}{5280}$.

Of course Fig. 27 does not show the playground itself; it is, in fact, merely the playground drawn to scale. Measure any side of it and see what scale has been used in the drawing.

and what the R.F. is. How long should the other sides of the figure be drawn? Have they the lengths they should have? Is the playground really drawn to scale?

(ii) All angles in the copy are the same size as the corresponding angles of the original. If this page is drawn to scale on a scale of 1" to 2" (i.e. R.F. = $\frac{1}{2}$), each side is halved in length, but the right angles are of course not halved; they remain right angles, as before.

Care should be taken to use a scale which is suitable. The drawing should not be too small. Also a scale such as 1 inch to 5 feet (or 10 feet) is much better than a scale such as 1 inch to 6 feet (or 8 feet) because the former leads to convenient decimals (1 foot = .2" or .1") and the latter leads to fractions or awkward decimals. Always use a ruler divided into tenths, as $\frac{1}{10}$ " or $\frac{1}{16}$ " is more convenient to use than $\frac{1}{8}$ " or $\frac{1}{4}$ ".

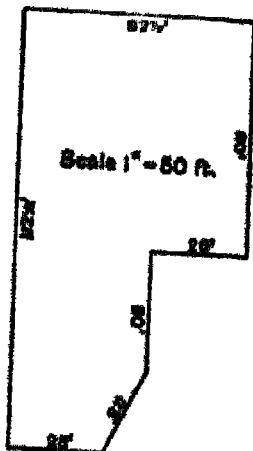


FIG. 27.

. In looking up to an object (X) (see Fig. 28), the eye has to be raised through an angle from the horizontal. This angle AOX is called the angle of elevation of X at O. Think of the angles of elevation of objects in the class-room as you view them when seated at your desk. So also when you look down at an object (Y) the angle AOY through which the eye has to drop from the horizontal in order to look at Y is the angle of depression of Y at O. Think of the angles of depression of the boys seated in your class-room as you view them when you are standing up.

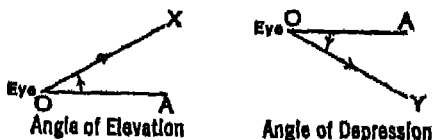


FIG. 28.

EXAMPLES 8.

(Drawing to Scale; Use of Instruments.)

1. A ship A is 6 miles North of a port B, and a ship C is 7 miles N.W. of A. Find the distance of C from B. (*Scale: 1 cm. to 1 mile.*)
2. A rectangular field is 150 yards long and 125 yards wide. Find the distance across from corner to corner. (*Scale: 1 inch = 50 yards.*)
3. A ladder 28 feet long rests against a wall with its foot 13 feet from the wall. How far up the wall does it reach? (*Scale: 1 inch = 10 feet.*)
4. A kite is flown by means of a string 280 feet long, and the string makes an angle of 35° with the ground. Find the height of the kite.
5. The elevation of the top of a tower is 40° at a point 110 yards away from its foot. Find the height of the tower.
6. A is 2.7 miles North of B, B is 3.5 miles East of C, and D is 1.4 miles South of C. How far is A from D? In what direction is A when viewed from D?
7. A boy ran after a rabbit, which started from A and ran, first of all 20 yards North, then 18 yards S.W., then 9 yards S. How far was it then from A in a straight line?

8. To reach the top of a mountain, I have to travel 3.4 miles up a slope of 25° and then 1.7 miles up a slope of 36° . How high is the mountain above the starting-point? (*The angles of a slope are reckoned with the horizontal.*)

9. X, Y, Z are three towns, Y being 10 miles due North of X. If Z bears S. 34° E. of Y and N. 71° E. of X, find the distances XZ, YZ.

10. A ship sails 4.8 miles in a straight line and then turns through 53° to the right, sailing afterwards 2.9 miles. Find the distance from the first to the last position.

11. The pendulum of a clock is 3 ft. 6 ins. long and it swings through 18° on each side of the vertical. Find the shortest distance between its two end positions.

12. The angle of depression of a ship at sea, when viewed from the top of a cliff 180 feet high, is 27° . Find the distance of the ship from the foot of the cliff.

13. A tower is 85 feet high and from the top are seen, in opposite directions, two men A and B, their angles of depression being 37° and 50° . Find the distance between A and B.

14. A and B are two consecutive milestones on a straight road, and C is an aeroplane just overhead at some point D between A and B; the angles of elevation of C are, 47° when viewed from A, and 62° when viewed from B. Find the height of the aeroplane (i) in miles; (ii) in yards.

15. AB is a straight line of coast 4 miles long, A being due East of B. C is a ship which bears N.E. of B and N. 25° W. of A. Find the distances of C from A and B.

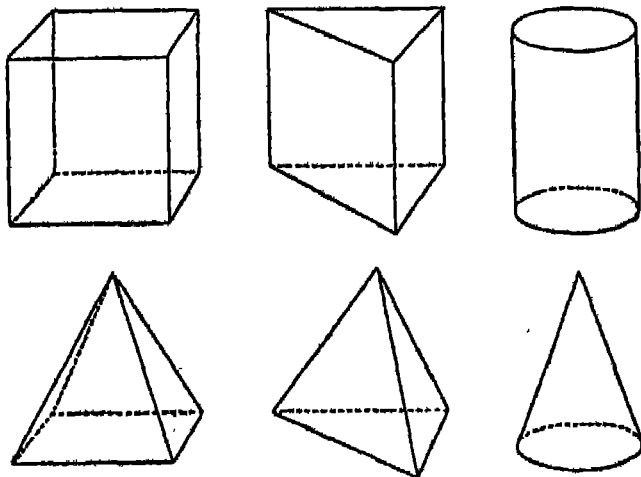
16. A tree is viewed from two points, X and Y, 25 yards apart, XYZ being a straight line, where Z is the foot of the tree. If the angles of elevation of the top of the tree at X and Y are 27° and 52° , find the height of the tree.

17. A, B, C are three towns; AB is 15 miles, and B is S.E. of A. C bears N. 38° E. of A and N. 17° W. of B. How far is C from A and from B?

18. A is a lighthouse placed 28 miles from a straight coast XY which runs North and South. The light can be seen at a distance of 86 miles. What length of coast can see the light?

19. Mount Everest is 29,000 feet high, and a member of the Everest Expedition is at a point A on the plateau of Tibet, which is 11,000 feet high at a distance from the summit of the mountain of 10 miles (52,800 feet). Draw a figure of straight lines by which the angle of elevation of the summit at A can be found, and measure the angle.

GEOMETRICAL SOLIDS.



The above sketches show the forms of the commonest Geometrical Solids, apart from the *sphere*. The first three figures, having a top and bottom which are identical and having vertical sides as well, are called *prisms*; the last three, tapering to a point, are called *pyramids*.

Of the *prisms* the first is called a *square* or *rectangular prism*, according as the base is a *square* or a *rectangle*; if all the edges are equal it is called a *cube*. The second prism, on a triangular base, is called a *triangular prism*; the third, on a circular base, is called a *circular prism* (or *cylinder*).

Of the *pyramids*, the first is called a *square* (or *rectangular*) *pyramid*, the second a *triangular pyramid* (or *tetrahedron*), the third a *circular pyramid* (or *cone*). A regular tetrahedron will have all its edges equal, and all its faces will be equal *equilateral* triangles.

Another well-known solid is the *sphere* (or *ball*). This is bounded by a curved surface, every point on which is at the same distance from a fixed point inside, called the

centre. If a plane surface cuts a sphere, it always meets it in a circle.

When the faces of a solid are all drawn on a plane surface, the figure produced is called the net of the solid. If the net is drawn on stiff paper or cardboard and bent along its lines it is possible to fold the figure so as to produce a hollow representation of the shape of the solid.

EXAMPLES 9.

(Many of these examples may well be taken at a much later stage of the course; they will be found useful in developing a pupil's sense of Solid Geometry.)

1. Fig. 29 shows the net of a cube. Draw such a net, each edge being $1\frac{1}{2}$ ", and then cut and fold the paper so as to form a cube.

2. Fig. 30 shows the net of a triangular prism. Draw this figure, making the long sides $2\frac{1}{2}$ " and the short ones $1\frac{1}{2}$ ", and fold it so as to form the prism.

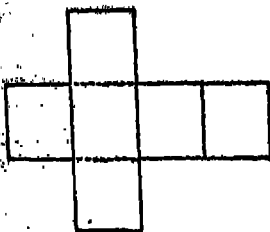


FIG. 29.

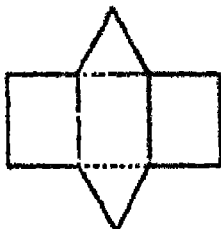


FIG. 30.

3. Draw the net of a square prism, the base edges being $1\frac{1}{2}$ " and the height $1\frac{1}{2}$ ".

4. Draw the net of a square pyramid, the base being a square on a side of $1\frac{1}{2}$ " and the slant edges being $1\frac{1}{2}$ ".

5. Draw nets of a rectangular prism and a rectangular pyramid. Mark the sides which are equal.

6. How many cubes of $1\frac{1}{2}$ " edge can be cut out of a cube of $3\frac{1}{2}$ " edge?

7. A cube is 1 yard each way and each edge is divided into 3 feet. It is then sawn up into cubes of 1 foot each way. How many such cubes are there? If the original cube was painted on the outside, how many of the smaller cubes have paint on (i) 3 faces; (ii) 2 faces; (iii) 1 face; (iv) no face at all?

8. $VABC$ is a regular tetrahedron, V being the vertex, and each edge is 1 inch long. If X bisects VA , and Y bisects VB , draw to scale the triangle VAB , and join XY . Measure the angles of $\triangle VAB$ and XY .

9. In Ex. 9 draw also the triangle CXY and measure the angle C .

10. V is the vertex and $ABCD$ is the square base of a pyramid, all of its sides being 4 cm. long; draw the $\triangle VAC$ and measure the angle V in it.

11. A square pyramid is drawn with its height 6 cm. and its base edges 4 cm. Draw to scale a triangle containing the height of the figure and the slant height as sides, and then draw one of the triangular faces of the pyramid to scale.

12. A sphere of radius 1" fits into a cubical box. Find the inside dimensions of the box.

13. Find the dimensions of a cylindrical box into which two spheres can be just put, one exactly above the other, the radius of each sphere being 4 cm.

14. Two cylinders 5" long and 1" in radius can be put side by side into a rectangular box and the lid will just fit. Find the dimensions of the box.

15. Find the size of a box on a square base which will just hold 36 cylindrical candles upright, each of length 8" and diameter $\frac{1}{2}$ ".

16. The radius of the base of a cone is 4 cm. and the slant edge is 7 cm. Draw a triangle to scale from which the height of the cone can be found. Measure the height.

17. The height of a cone is 6.5 cm. and the diameter of the base is 5 cm. Find by drawing (i) the slant height of the cone, and (ii) its vertical angle.

18. How many marbles, each $\frac{1}{2}$ " in diameter, can be packed in a rectangular box which is $4\frac{1}{2}$ " long, 3" broad, and 2" deep inside?

19. A cubical box fits into a hollow cylinder, with four of its edges lying along the curved surface. The diameter of the cylinder is 4". By drawing the base of the cylinder find the length of an edge of the box.

20. Find, by drawing and measuring, the radius of a hollow sphere inside which a cube of edge 3" can be put exactly.

STAGE II.

FUNDAMENTAL TRUTHS.

ANGLES AT A POINT.

Definitions. Adjacent angles (i.e., angles next to each other) are angles that have a common arm and lie on opposite sides of that arm. Thus in Fig. 31 the angles $\angle AOB$, $\angle BOC$ are adjacent, OB being the common arm; also in Fig. 35 the angles $\angle AOB$, $\angle BOC$ are adjacent.

EXAMPLES 10.

1. Name other pairs of adjacent angles in Figs. 35, 36, 37.

Notice that the outer arms of the angles may be in a straight line (such as BO and CO in Fig. 31) or in a broken line (such as CO and OA in Fig. 35).

2. Draw several pairs of adjacent angles with their outer arms in these two positions—(a) straight, (b) not straight.

Two angles whose sum is one right angle are said to be complementary.

Two angles whose sum is two right angles are said to be supplementary.

Thus the angles 30° and 60° are complementary ($30^\circ + 60^\circ = 90^\circ$); also the angles 35° and 145° are supplementary ($35^\circ + 145^\circ = 180^\circ$).

3. Name in degrees and in right angles several pairs of complementary angles; also several pairs of supplementary angles.

4. What are the complements of (i) 30° ; (ii) 25° ; (iii) 15° ?

5. What are the supplements of (i) 40° ; (ii) 105° ; (iii) 25° ?

6. Draw two complementary adjacent angles. Label each in degrees.

7. Draw two supplementary adjacent angles. Label each in degrees.

8. Fig. 31 shows a straight line AO standing on another straight line BOC . Draw a similar figure, making the angles of any size you please. Use your protractor to measure the adjacent angles AOB , AOC . (This can be done without moving the protractor, as in Fig. 32.) Find the sum of the angles AOB , AOC .

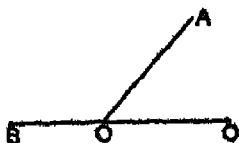


FIG. 31.

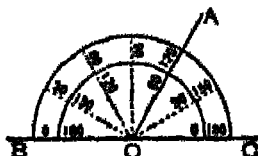


FIG. 32.

9. In Fig. 32, using the figures shown, say what is the size of
 (i) the $\angle BOA$,
 (ii) the $\angle AOC$,
 (iii) the $\angle BOA +$ the $\angle AOC$.

What is the sum of the $\angle$ s BOA , AOC ? How many right angles? Will this always be so for other positions of OA ?

10. Consider AO to be a sort of hand moving like a clock-hand between CO and BO . Does the protractor tell you the size of the one angle (BOA) if the other angle (AOC) is known?

What will *always* be the sum of these adjacent angles?

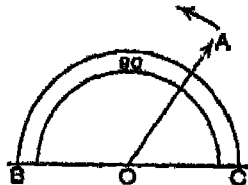


FIG. 33.

11. In Fig. 31

find $\angle BOA$, if $\angle AOC =$ (i) 70° , (ii) 65° , (iii) x° ;
 find $\angle AOC$, if $\angle BOA =$ (i) 100° , (ii) 150° , (iii) y° .

12. XOY is an angle and XO is continued (or produced) to Z . (i) If $\angle XOY$ is known, how can you find $\angle YOZ$? (ii) What is $\angle XOY + \angle YOZ$? (iii) If $\angle YOZ$ is twice $\angle XOY$, find the size of each.

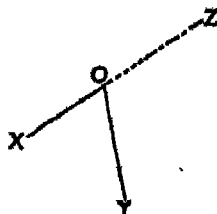


FIG. 34.

13. Draw a straight line CD , and from a point O in it draw any straight lines OA , OB (both above CD). Measure the

angles COB, BOA, AOD and add them together. Account for the result.

If some boy obtains 179° , what would you think of his result?

If another boy obtains 185° , what would you think of his result? How do you account for it?

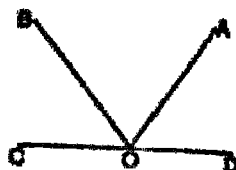


FIG. 34.

14. If in Fig. 34, $\angle COB = 70^\circ$, $\angle BOA = 85^\circ$, find $\angle AOD$

(i) by drawing a figure; (ii) without drawing a figure.

15. If in Fig. 34, $\angle COB = 53^\circ$, $\angle AOD = 46^\circ$, calculate $\angle BOA$.

16. What is the sum of the three angles in Fig. 34 in every case?

17. What is the sum of the four angles in Fig. 34? (Give answer at sight, and check your result by drawing a similar figure and measuring your four angles by means of a protractor.)

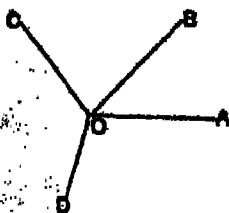


FIG. 36.

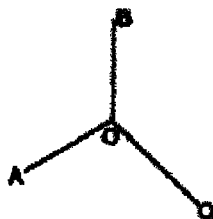


FIG. 37.

18. What is the sum of the three angles in Fig. 37? If $\angle BOA = 119^\circ$, $\angle BOC = 103^\circ$, calculate $\angle COA$.

19. If all 3 angles are equal in Fig. 37, find the size of each.

20. Are all adjacent angles supplementary? (Draw figures.)

21. Are all supplementary angles adjacent? (Draw figures.)

In Figs. 8-10 we learn an important Fact which can be stated thus:—

TRUTH 1.

If a straight line stands on another straight line, the sum of the adjacent angles so formed is equal to two right angles.

22. What is the sum of the four angles at O if AOC is straight? Also if AOC is not straight. (See Fig. 38.)

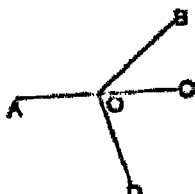


FIG. 38.

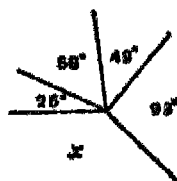


FIG. 39.

23. In Fig. 39 calculate x in degrees. Check by drawing to scale.
 24. A wheel has 8 spokes, spaced evenly round the rim. What is the angle between two adjacent spokes?
 25. Suppose that angles of 40° , 50° , 60° , 70° , 110° , 130° , 180° , 140° were cut out on separate pieces of paper and given to you. Name several pairs of these angles which could be placed back-to-back, with their outer arms in a straight line (as in Fig. 40).

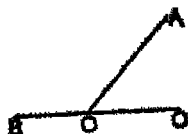


FIG. 40.

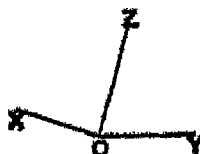


FIG. 41.

26. Draw a straight line OZ. On one side of it make an angle $ZOY = 76^\circ$, and on the other side make $ZOX = 100^\circ$. Are XO and OY in a straight line? If not, is XO above or below the produced part of OY? If ZOY is still 76° , how big would ZOX have to be if XO and OY were in a straight line? (See Fig. 41.)
 27. If two adjacent angles (see Fig. 40) have their exterior arms (BO and OC) in the same straight line, what must be their sum? This suggests TRUTH 2, which is stated thus:

If two adjacent angles are together equal to two right angles, the exterior arms of the angles are in the same straight line.

28. In Fig. 42, $\angle AOB = x^\circ$, $\angle BOC = 2x^\circ$, reflex $\angle COA = 5x^\circ$; find x and the size of the three angles. Check by adding.

29. OA, OB, OC are three lines from O so that $\angle AOB = 72^\circ$, and $\angle BOC = 84^\circ$. OP bisects $\angle AOB$ and OQ bisects $\angle BOC$; find $\angle POQ$.

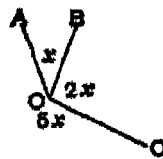


FIG. 42.

VERTICALLY OPPOSITE ANGLES.

30. In the figure, $\angle AOC = 65^\circ$; calculate the other three angles by means of TRUTH 1. What angle is equal to $\angle AOC$? Are any other angles equal? Verify by a drawing to scale.

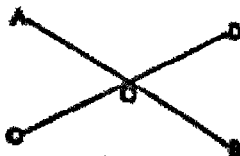


FIG. 43.

31. In Fig. 43 show that the angles are equal in pairs, (i) if $\angle AOC = 45^\circ$; (ii) if $\angle AOC = 120^\circ$; (iii) if $\angle AOC = x^\circ$. Will these angles be always equal in pairs?

32. Hold a stick in your hand, pointing East and West, and let your hand be at O. Turn your stick until it points N.E. and S.W. What angle has the left-hand part turned through? What angle has the right-hand part turned through? What conclusion do we reach? Notice that we have learnt the same fact in Exs. 30, 31. We state it as

TRUTH 3.

If two straight lines intersect, the vertically opposite angles are equal.

Definition. When two straight lines intersect, the opposite pairs of angles are called vertically opposite.

Thus in Fig. 43, $\angle AOC$ and $\angle BOD$ are vertically opposite angles; name another pair.

33. If in Fig. 44 $\angle a = 40^\circ$, $\angle p = 125^\circ$, $\angle x = 85^\circ$, find the size of all the other angles, giving reasons.



FIG. 44.

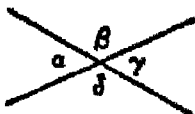


FIG. 45.

34. In Fig. 44 there are several pairs of equal angles; state which they are, and give reasons.

35. If in Fig. 45 $a + \beta + \gamma = 260^\circ$, calculate the size of each of the four angles.

36. If in Fig. 45 $\beta = 40^\circ$, calculate each of the four angles.

(Other examples on angles at a point will be found on page 30.)

DIRECTIONS OF LINES. PARALLEL LINES.

EXAMPLES II. (Oral.)

1. Several boys in a room point at another boy. Are they pointing in the same direction? If they pointed at the same nail on the wall, are they pointing in the same direction?

2. (a) If several boys in the playground point at a church tower a mile away, are they actually pointing in the same direction? Are they pointing in nearly the same direction?

(b) If they all point East, are they pointing exactly in the same direction? Notice the difference between (a) and (b).

Another name for *straight lines in the same direction* is *parallel lines*. Notice that they can never meet.



FIG. 46.

[Lines which are nearly in the same direction meet a long way off.]

The smaller the angle between two lines, the nearer they are to one another in direction.

The larger the angle between two lines, the more apart their directions are.

We have also to understand what is meant by *skew lines*.

Skew lines are lines that are not in the same plane; they can therefore never meet.

Thus in the box shown the lines AB and CD are skew.

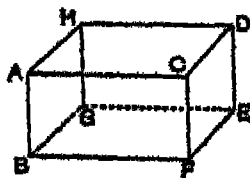


FIG. 47.

3. Point out other pairs of skew lines in Fig. 47.

4. Point out pairs of skew lines in the classroom.

5. Hold out your ruler and your pencil so that their directions represent a pair of skew lines.

6. In how many directions can a person travel along a straight line? How many degrees will there be in the angle formed by two lines that run S. and N.W.?

7. Through how many degrees must I turn a line that faces N. until it faces S.W., if I turn it (i) anti-clockwise; (ii) clockwise?

8. A line runs due E.; I turn it clockwise through an angle of 45° . In what direction does it point? Does the answer depend at all on the particular point about which I rotated the line?

9. Draw two parallel lines on your paper by means of the top and bottom of your ruler; produce them both ways until they reach the sides of the paper. Are the lines any further away from each other at one place than at another? Would they ever meet?

10. Do skew lines ever meet? Would it do to say that lines that never meet are parallel?

11. Hold out two pencils to indicate two lines which are in the same plane and which never meet, however far they are produced. Are the lines parallel?

We now have a second definition of parallel lines:

Parallel lines are lines which are in the same plane, and which never meet, however far they are produced.

In this book we shall use the first definition of parallel lines. (See page 29.)

TRUTHS ABOUT PARALLEL LINES.

EXAMPLES 12.

1. Two boys X and Y are walking in a Northerly direction a few yards apart in the same playground. When a bell rings they have both reached a narrow asphalt path BCDE as shown, which runs S.W. and N.E., leads to the school door at E, and so they turn and walk along it. What angle has X turned through at C? What angle has Y turned through at D?

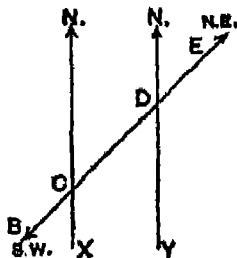


Fig. 48.

2. I am walking from A along a path, represented by ABCF, which runs into a wide road with parallel pathways BE and CD. If I turn at B into a pathway, what angle do I turn through? If I go on to C before turning, what angle do I turn through? What conclusion follows from this? (See Fig. 49.)

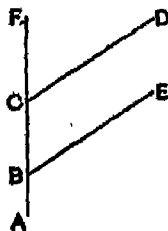


Fig. 49.

3. Several boys are standing in single file behind each other. They all turn through 40° to the right. Do they now face in the same direction? Illustrate by means of a figure.

4. AB and CD are parallel lines (as in Fig. 50) and are therefore in the same direction (say E. and W.). EF is a third line cutting them at G and H. Suppose its direction from N.E. to S.W. What is the size of (i) $\angle EGB$; (ii) $\angle GHD$? Are they always equal as long as AB and CD are in the same direction?

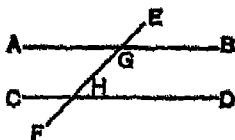
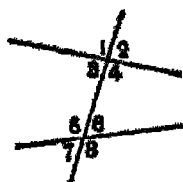


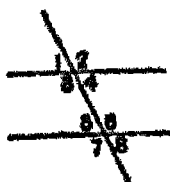
Fig. 50.

5. Put down your pencil in the position EF; turn it about G until it lies in the direction BA. Also turn it from EF about H until it lies in the direction of DC. What angles has it turned through in the two cases? As BA and DC are in the same direction, what conclusion follows? (See Fig. 50.)

6. Draw the figure to Ex. 4 by making AB and CD correspond to the top and bottom of your ruler. Will AB and CD be parallel? Now draw EF across them at any acute angle (say 47°). Which are the angles of 47° ?



Non-parallel lines met by a transversal.
Fig. 1.



Parallel lines met by a transversal.
Fig. 2.

FIG. 31.

In the above figures, 2 and 6 are called corresponding angles.

[There are other pairs of corresponding angles, such as 3 and 7; name two other pairs.]

Corresponding angles always form a sort of letter *V* or *F*, though the arms need not be parallel.

7. If $\angle 2 = 40^\circ$, write down the values of the various angles as far as possible in Figs. 1 and 2 above.

We have thus learnt our *first Truth* about Parallel Lines:

* If a pair of corresponding angles are equal, the lines must be parallel.

In the above figures, 3 and 6 (forming a sort of letter *Z* or *3*) are called alternate angles. There is another pair of alternate angles (4 and 5).

8. If in Fig. 2 $\angle 3 = \angle 6$, why does $\angle 2 = \angle 6$? What can you say about the lines which are met by the transversal?

9. I walk in a straight line (AB), turn 40° to the right (i.e. face BC), walk as far as C, and then turn 40° to the left. What can you say about my first and last directions? Illustrate by a figure.

10. A ship is sailing due North, and then changes her course by 25° towards the East. Later on she changes her course by 25° towards the West. In what direction is she now sailing? What relation is there between this direction and her first direction?



FIG. 33.

* For fuller statements on this Truth and on the Truths on page 33, refer to the enunciations on pages 61-64.

11. A framework consists of two parallel sides linked together by horizontal wooden bars between the sides. If one of the bars makes an angle of 42° with the left-hand side, what angle does it make with the right-hand side?



FIG. 52.

We now learn our *second Truth* about parallel lines:—

If a pair of alternate angles are equal, the lines must be parallel.

12. Hold a pencil at B as shown, and turn it clockwise through an angle of 70° (marked p). In what direction does the pencil now point? Through what angle must it now be turned clockwise for it to lie in the opposite direction to its original direction? What will be the size of the angles q and r ?

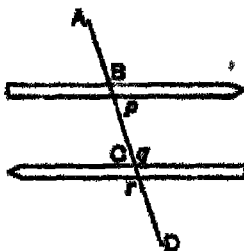


FIG. 53.

13. Examine Fig. 51 (ii) again. If angle 4 is 35° and the lines are parallel, label all the other angles in degrees. What is the size of angle 6? What do you know about angle 4 + angle 6?

14. Suppose in Fig. 51 (ii) that $\angle 4 + \angle 6 = 180^\circ$. What can be said about (i) angles 2 and 4; (ii) angles 2 and 6; (iii) the lines met by the transversal?

We therefore learn a third Truth about parallels:—

If a pair of interior angles on the same side of the cutting line make up two right angles, the lines must be parallel.

We can of course say as well:

If two lines are parallel—

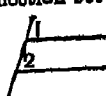
- (i) the corresponding angles are equal;
 - (ii) the alternate angles are equal;
- and (iii) two interior angles on the same side of the cutting line are equal to two right angles.

15. Consider Fig. 51 (ii) again, and suppose that angle 7 = 62° . If the lines are parallel, determine the other seven angles, giving reasons.

(SHORTER)

D

16. The figures below show a line cutting two parallel lines. Find the connection between the sizes of the angles marked.



(i)



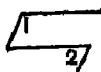
(ii)



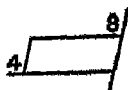
(iii)

FIG. 56.

17. The figures below contain parallel lines (in pairs). Find the relation between the angles marked.



(i)



(ii)



(iii)

FIG. 57.

18. BE and CF are parallel and BE is perpendicular to AD; prove that CF is also perpendicular to AD. (See Fig. 57.)

19. In Fig. 58 AB and XY are parallel, and angle 1 = angle 2; show that angle 3 must equal angle 4.

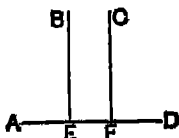


FIG. 57.

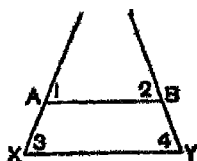


FIG. 58.

20. A four-sided figure ABCD has its opposite sides parallel.

- If angle A = 70° , calculate the other angles, giving reasons.
- Show that the angles A and C are always equal.

21. AB and CD are parallel (Fig. 59), and so are EF and GH; GF is drawn, and angle $a = 40^\circ$ and angle $b = 32^\circ$. Find the values of the angles p, q, r, s, t, u, v , giving your reasons.

22. If in Fig. 59 $a = b$, prove that $r = s$.

Show that $q + v = 180^\circ$ always; also that $a + r = p$.

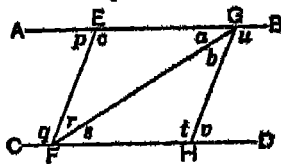


FIG. 59.

Definitions.

The following definitions are placed together for reference :

A rectilinear figure is a plane figure that is bounded by *straight lines*.

A triangle is a plane figure bounded by *three straight lines*.

In the $\triangle ABC$, AB , BC , CA are called the *sides*; the points A , B , C are called the *angular points* or *vertices*.

If we think of one corner as vertex, say B , then AC , the opposite side, is said to be the *base*.

The names of its *angles* are A (or BAC or CAB), B (or ABC or CBA), C (or ACB or BCA). The small letters, a , b , c are sometimes used to denote the sides; in that case a is BC , the side opposite A . Which is b ? and c ?



FIG. 60.

If all three sides of a triangle are equal, it is said to be *equilateral*.

If only two sides are equal, it is said to be *isosceles*.

If no two sides are equal, it is said to be *scalene*.

In an isosceles triangle, the unequal side is generally taken as the *base*, and the opposite angle as the *vertex*.

If a triangle has one of its angles a *right angle*, it is said to be *right-angled*, and the side *opposite the right angle* is called the *hypotenuse*.

A quadrilateral is a plane figure bounded by *four straight lines*.

Either line joining two opposite corners is said to be a *diagonal*.

A polygon is a plane figure bounded by *more than four straight lines*.



FIG. 61.

Certain polygons have special names:

If with 5 sides, it is called a *pentagon*.

If with 6 sides, it is called a *hexagon*.

If with 8 sides, it is called an *octagon*.

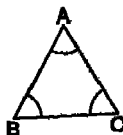
If with 10 sides, it is called a *decagon*.

The *perimeter* of any figure is the sum total of all its sides.

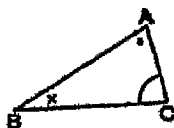
A rectilinear figure is said to be:

- (i) equilateral when all its *sides* are equal,
- (ii) equiangular when all its *angles* are equal,
- (iii) regular when it is both equilateral and equiangular.

There is another meaning of the word *equiangular*. If two triangles ABC , DEF are so related that $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$, they are said to be equiangular (to one another), and the same for other figures. There need not be any confusion: an *equiangular triangle* has its own angles equal to one another; two *equiangular triangles* have the angles of the first equal in turn to the angles of the second.



(i)



(ii)



(iii)

An equilateral triangle.

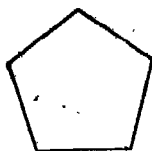
$$\angle A = \angle B = \angle C.$$

Two equiangular triangles.

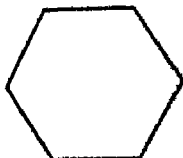
$$\angle A = \angle D, \angle B = \angle E, \angle C = \angle F.$$

FIG. 62.

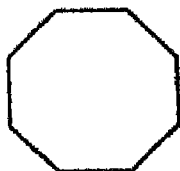
Later on it will be seen that an *equiangular triangle* (see Fig. 62 (i)), is also *equilateral*.



Regular Pentagon



Regular Hexagon



Regular Octagon



Irregular Pentagon



Irregular Hexagon



Irregular Octagon

FIG. 63.

A *convex polygon* is one in which every angle is less than two right angles.

A *concave polygon* is one in which at least one of the angles is reflex.

THE ANGLES OF A TRIANGLE.

EXAMPLES 13.

1. Draw any large triangle ABC. Measure its angles and find their sum.

Compare your answer with the answers obtained by other pupils. What is probably the correct answer?

2. Do the same exercise again with different triangles. What do you think the sum of the three angles of a triangle really is? What would you tell a pupil who obtained the answer as (i) 179° ; (ii) 160° ?

You may be even more sure of the result after working through the following exercises.

3. Draw a triangle on paper and cut it out; tear off the three angles and put them together again with the three angles at the same point. What do you notice? What does this seem to show?

4. Draw a triangle ABC and place your pencil along BC, with the point towards C. Turn the pencil anti-clockwise about B until it lies along BA; then about A until it lies along CA; then about C until it lies along CB. What three angles has it turned through in turn? What connection is there between its first direction and its last? What total angle has the pencil turned through:



FIG. 64.

(i) in terms of angles of the triangle;

(ii) in degrees or right angles?

What conclusion follows from this experiment?

5. Draw a triangle ABC and produce BC to D. The line CE in the figure is drawn parallel to BA. Name a pair of alternate angles and a pair of corresponding angles. What angle is equal to (i) $\angle A$; (ii) $\angle B$; (iii) $\angle A + \angle B$? (Give reasons.) What two angles at C are together equal to the three angles of the triangle? What fact does this illustrate?

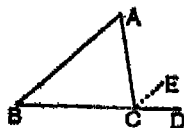


FIG. 65.

All the above exercises seem to point to the important fact:

The sum of the angles of a triangle is equal to two right angles.

6. If in a triangle ABC, $\angle A = 43^\circ$, $\angle B = 85^\circ$, calculate $\angle C$. Check by a figure drawn to scale.

7. If two angles of a triangle are 73° and 41° , what is the third angle?

8. If in a right-angled triangle one angle is 49° , find the third angle.

9. If in a triangle ABC , $\angle A = \angle B$ and $\angle C = 50^\circ$, calculate A and B .

10. If in Fig. 65 $\angle ACD = 118^\circ$ and $\angle B = 47^\circ$, find $\angle A$ and $\angle ACB$.

11. If in Fig. 65 $\angle A = \angle B$ and $\angle ACD = 110^\circ$, calculate all the angles of the triangle.

12. If the angles of a triangle are all equal, express each (i) in degrees; (ii) as a fraction of a right angle.

13. The angles of a triangle are x° , $4x^\circ$, $5x^\circ$; find x .

14. Can a triangle be drawn with angles (i) 51° , 47° , 82° ; (ii) 23° , 85° , 82° ?

15. If in a triangle $\angle A$ is 4° larger than $\angle B$, and $\angle C$ is 32° smaller than $\angle B$, calculate all the angles.

16. In a triangle ABC , $\angle A = 84^\circ$, $\angle B = 78^\circ$, AD bisects $\angle A$ and meets BC in D ; BE is perpendicular to AC . Draw a rough figure and find each angle in the figure by calculation.

17. In $\triangle ABC$, $\angle A = 24^\circ$, $\angle C = 21^\circ$; CX is perpendicular to AB produced. Show that $\angle XBC = \angle XCB$.

18. Cut out in paper any triangle such as ABC , and suppose that AD is perpendicular to BC . Show, by creasing the paper along the dotted lines, that A , B , C can be folded over on to D . Notice how the lines fall at D . What fact does this illustrate?

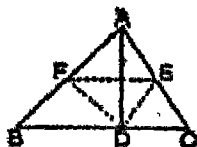


FIG. 65.

19. Show, by drawing figures, when the $\angle A$ of a triangle ABC is (i) equal to $\angle B + \angle C$; (ii) greater than $\angle B + \angle C$; (iii) less than $\angle B + \angle C$.

20. Draw a triangle ABC on a base BC 1 inch long, having $\angle B =$ twice $\angle A$, and $\angle C =$ three times $\angle A$.

21. Sketch the following kinds of triangle:

- (i) an equilateral triangle;
- (ii) a right-angled isosceles triangle;
- (iii) an obtuse-angled isosceles triangle;
- (iv) an acute-angled scalene triangle.

22. Draw a right-angled triangle. What kind of angle is each of the other two angles in the triangle? Give a reason.

23. What is the least number of acute angles in a triangle? What is the greatest number?

ANGLES OF A QUADRILATERAL AND POLYGON.

EXAMPLES 14.

1. Draw a large quadrilateral $ABCD$; measure all four angles and find their sum. Work out the sum again in a quadrilateral of different size and shape. Compare your two answers. Have you any idea what the answer should be?

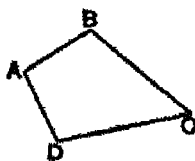


FIG. 57.

2. Draw a quadrilateral $ABCD$ and join BD ; without using your protractor, state what is the sum of:

- (i) all the angles of $\triangle ABD$;
- (ii) all the angles of $\triangle CBD$;
- (iii) all the angles of the quadrilateral $ABCD$.

3. Using the result of Ex. 2, find the fourth angle of a four-sided figure if three angles are 49° , 85° , 127° . Check with a figure drawn to scale.

4. If the four angles of a quadrilateral are equal, express each in degrees. Draw two such quadrilaterals quite different in shape.

5. If one angle of a quadrilateral is 93° , and the other three are equal, find the size of each.

6. Draw a pentagon and divide it into triangles as shown. What is the sum of



FIG. 58.

- (i) all the angles of all the triangles;

- (ii) all the angles of the pentagon?

7. By dividing a hexagon into triangles, find the sum of all its angles.

COPYING A GIVEN TRIANGLE.

In order to find out how to copy a given triangle ABC , we may have to use the sides of the figure, or the angles, or both. The angles are, of course, A , B , C . The sides are often called a , b , c after the names of the angles to which they are opposite (see figure).

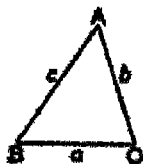


FIG. 59.

In copying a triangle it is always a good thing to draw a small sketch of the triangle freehand and insert the numbers corresponding to the sides and angles given.

How many of the six *elements* named above (3 *sides* and 3 *angles*) must be known before we can copy the triangle?

Do we need to know all six?

Is one element sufficient (one side or one angle)?

Are two elements sufficient (two sides or two angles or one side and one angle)?

Now let us try three elements.

METHOD I of copying a Triangle.

To draw a triangle, given "two sides and the included angle."

Suppose that $AB = 2''$, $AC = 1.5''$,
 $\angle A = 50^\circ$.

Can we draw the triangle from these figures? How?

Evidently $\triangle ABC$ can be drawn from the three given elements—two sides and the included angle.

How many different triangles ABC can be drawn if the drawings are all correct? If you are not sure, draw another $\triangle DEF$ having $DE = 2''$, $DF = 1.5''$, $\angle D = 50^\circ$. Measure $\angle s B$ and E ; $\angle s C$ and F ; bases BC and EF . Do they agree?

If two triangles agree as regards two sides and the included angle, what do you know about them?

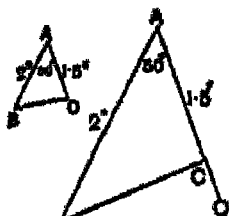


FIG. 70.

EXAMPLES 15.

1. Draw a triangle ABC of any size you please, and copy it, using two sides and the included angle (BA , BC , $\angle B$). Measure the other sides and angles of the two triangles. What can you say about the two triangles? (Do not draw the sides of the new triangle nearly parallel to the sides of the old.)

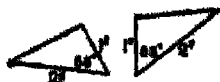


FIG. 71.

2. Draw two triangles with the sides and angles given in Fig. 71. Do they agree in all their elements?

We therefore see that a triangle can be drawn in shape and size completely when two sides and the included angle are known.

This is the same as saying:

Two triangles which agree in two sides and the included angle agree in all other respects. Such triangles are said to be congruent.

[For examples on *two sides and included angle* see pages 45 and 72.]

METHOD II of copying a Triangle.

To draw a triangle, given "two angles and one side."

Suppose that $\angle B = 64^\circ$, $\angle C = 48^\circ$,
 $BC = 2.5''$.

Can the triangle be copied? How?

3. If two Δ s ABC, DEF are drawn so that they agree as regards two angles and a side ($\angle B = \angle E = 61^\circ$, $\angle C = \angle F = 108^\circ$, $BC = EF = 6$ cm.), do they agree in all other particulars? Draw them and measure.

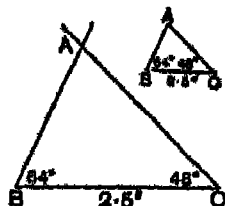


FIG. 72.

4. Draw a triangle of any size you please, and copy it using two angles and a side between them; measure and see if their other sides and angles are equal.

5. Can a triangle ABC be copied if $BC = 2''$, $\angle B = 40^\circ$, $\angle A = 75^\circ$? What is the size of $\angle C$? Copy the triangle.

We therefore see that a triangle can be drawn in size and shape completely when two angles and a side are known.

This is the same as saying:

Two triangles which agree in two angles and a side are congruent.

Note that the sides which are equal must *correspond* in the two triangles; for instance, if in the triangles ABC, DEF, $\angle B = \angle E$, $\angle C = \angle F$, $BC = EF$ (or $AB = DE$), the triangles are congruent. If, however, $\angle B = \angle E$, $\angle C = \angle F$, but BC (opposite $\angle A$) $\neq DF$ (opposite $\angle E$) the triangles are not congruent.

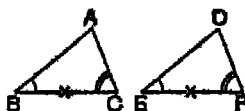


FIG. 73.



FIG. 74.

[For examples on *two angles and one side* see pages 45 and 72.]

METHOD III of copying a Triangle.

To draw a triangle, given "three sides."

Suppose that $BC = 2.4''$, $CA = 2.1''$, $AB = 3''$.

Can we draw a triangle having these sides? First draw BC . How far is A from B ? Then if we draw a circle with centre B and radius $3''$, A will lie somewhere on it.

How far is A from C ?

Then if we draw a circle with centre C and radius $2.1''$, A will lie somewhere on it.

A must therefore lie where these circles intersect.

Can more than one triangle be drawn on the base BC having the sides as above? The complete circles meet in two points, say A and A' .

Make the two Δ s ABC , $A'BC$ on paper, cut them out and convince yourself that the two triangles are congruent.

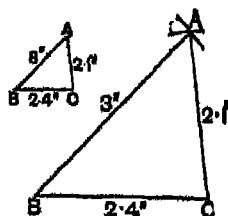


FIG. 75.

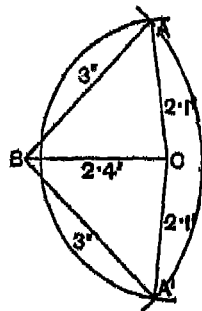


FIG. 76.

5. Draw two triangles as shown in Fig. 77 with sides 5 cm., 4 cm., 6 cm., cut them out and show that they really can be made to fit on each other.

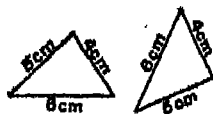


FIG. 77.

We therefore see that a triangle can be drawn in size and shape completely when the three sides are known.

This is the same as saying:

Two triangles which agree in the three sides are congruent.

[For examples on three sides see pages 45 and 79.]

We have been copying a triangle from three elements. There are other sets of three elements to consider.

IV. Three Angles.

V. Two sides and a non-Included angle.

IV. Three Angles of a Triangle.

Suppose that $\angle A = 50^\circ$, $\angle B = 72^\circ$, $\angle C = 58^\circ$.

Can more than one triangle be drawn having these angles?

Evidently, yes, more than one can be drawn, for no side is given and the base can be drawn of any size. (See Figures.)

Can anything be said about the many triangles that can be drawn? Are they not exact copies of each other on different scales? Then the sides will not be equal; but they will be proportional.

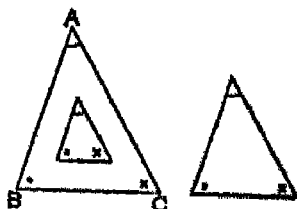


FIG. 73.

In fact they have the same shape, but have different sizes.

When one of the sides is known, the triangle is known completely, but the copying of it now comes under Case II, "Two angles and a side"; if two angles are known, the third is known as well.

6. Draw a triangle with a base of 2"; measure its angles and draw another triangle on a base of 1" having the same two base angles as the first one. Measure the sides of both triangles. Are the sides of the first triangle double the corresponding sides of the second?

The triangles then (i) have the same shape;
(ii) are different in size.

7. Draw one triangle on a base of 2" and another on a base of 8 cm., with the base angles of the first equal to the base angles of the second. Measure the other sides of the first in inches and those of the second in centimetres. The base of the second represents the base of the first on a scale of 4 cm. to 1 inch. Is this true (or nearly true) of the other sides? If so, the second triangle is a copy of the first but on a different scale. Again we see that (i) the triangles agree in shape; (ii) the triangles do not agree in size.

V. Two Sides and a non-included Angle.

Suppose that $AB = 2''$, $AC = 1.6''$, $\angle B = 40^\circ$.

Can we draw one triangle only from these figures?

First draw AB , then make $\angle B$, leaving the arm BD not definite in length. Do we know the length of BC ?

How then shall we fix C ?

How far is C from A ? Then C lies on a circle whose centre is A and whose radius is $1.6''$. The figure shows part of this circle and shows it cutting the base BD in two points, C_1 and C_2 .

We have now made two triangles with the given measurements, ABC_1 and ABC_2 .

The triangles (see Fig. 80) can be separated and placed in different figures with different letters. They are evidently not congruent.

We see, then, that two triangles need not be congruent even if they agree in Two Sides and a non-included Angle.

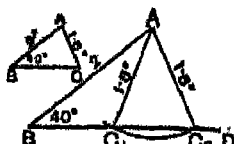


FIG. 79.



FIG. 80.

8. Draw a triangle ABC given that $BC = 5$ cm., $BA = 7.5$ cm., $\angle C = 125^\circ$. Can more than one triangle be drawn? Is there any chance of two different triangles being drawn from two sides and a non-included angle if the given angle is obtuse?

9. Draw a ΔABC given that $CA = 4$ cm., $CB = 7$ cm., $\angle A = 90^\circ$. Can more than one triangle be drawn? If two are drawn in the same figure, show by measuring that they agree in other particulars.

We note then that two triangles are congruent if they agree in two sides and non-included angle, provided the given angle is a right angle or an obtuse angle.

N.B.—In applying the tests for Congruent Triangles, it is important to see whether the sides and angles taken correspond to one another. When angles are taken first, corresponding sides are those which are opposite to equal angles; when sides are taken first, corresponding angles are those which are opposite to equal sides.

10. In a triangle ABC angle A is a right angle and AD is drawn at right angles to BC . If $\angle B$ is 30° , calculate all the other angles of the figure.

The $\Delta s ABD, ACD$ now have

$\angle B = \angle CAD (= 30^\circ)$, $\angle ADB = \angle ADC (= 90^\circ)$, side AD is common.

Does this prove the triangles congruent by *two angles and a side*? Examine carefully whether the side AD corresponds in the two triangles, i.e., whether it is opposite equal angles. (Which angle is it opposite? Are these angles equal?)

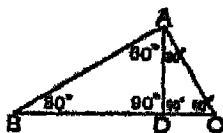


FIG. 81.

11. In Ex. 10 consider the $\Delta s ABD, ABC$. Show that they have two angles of one = two angles of the other, and a side common; does this side correspond? Are the triangles congruent?

Notice that "*two angles and a side*" if rightly used means "*two angles of one triangle = two angles of the other, and a side of one triangle = the corresponding side of the other.*"

12. Take care also that in "*two sides and the included angle*" the angle taken is always included by the sides already taken. ABC is an isosceles triangle, having $AB = AC$ and $\angle B = \angle C$. D is any point in BC except the middle point.

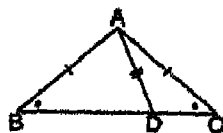


FIG. 82.

Show that $\Delta s ABD, ACD$ agree in two sides and a corresponding angle, but that they are not congruent. Why not?

EASY EXERCISES ON CONGRUENT TRIANGLES.

EXAMPLES 16.

Draw rough figures of the pair of triangles ABC, PQR in the following cases; label the sides and angles with the numbers given. Is there enough evidence given in each case to show that the triangles are congruent? (Give reasons for your answer.) If they are congruent, write down under your pairs of triangles which other sides and angles are equal.

1. (i) $AB = 3''$, $BC = 4''$, $\angle B = 70^\circ$; (ii) $PQ = 3''$, $QR = 4''$, $\angle Q = 70^\circ$.

2. (i) $\angle A = 40^\circ$, $AB = 6$ cm., $\angle B = 50^\circ$; (ii) $\angle P = 40^\circ$, $PR = 6$ cm., $\angle R = 50^\circ$.

3. (i) $AB = 4''$, $BC = 5''$, $CA = 6''$; (ii) $PQ = 6''$, $QR = 5''$, $RP = 4''$.

4. (i) $\angle A = 55^\circ$, $\angle B = 34^\circ$, $AB = 3\frac{1}{2}''$; (ii) $\angle P = 34^\circ$, $\angle Q = 91^\circ$, $PR = 3\frac{1}{2}''$.

5. (i) $AC = 5''$, $AB = 7''$, $\angle A = 35^\circ$; (ii) $PR = 5''$, $PQ = 7''$, $\angle R = 35^\circ$.

6. (i) $AC = 1''$, $BC = 2''$, $\angle B = 20^\circ$; (ii) $PR = 1''$, $QR = 2''$, $\angle Q = 20^\circ$.

7. (i) $AB = 2''$, $BC = 2\frac{1}{2}''$, $CA = 3\frac{1}{2}''$; (ii) $PQ = 2''$, $QR = 3\frac{1}{2}''$, $RP = 2\frac{1}{2}''$.

8. (i) $\angle A = 111^\circ$, $\angle C = 31^\circ$, $BC = 2''$; (ii) $PQ = 2''$, $\angle Q = 31^\circ$, $\angle P = 38^\circ$.

9. (i) $\angle A = 105^\circ$, $AB = 5''$, $BC = 6''$; (ii) $\angle P = 105^\circ$, $PQ = 5''$, $QR = 6''$.

Are the Δ s ABC, XYZ congruent in the following cases? (Draw freehand figures to illustrate answers, and give reasons for your answers.)

10. $AB = XY$, $BC = YZ$, $\angle B = 90^\circ = \angle Y$.

11. $AB = XY$, $\angle A = 90^\circ = \angle X$, $BC = YZ$.

12. $AB = YZ$, $\angle A = \angle Y$, $\angle C = \angle Y$.

13. $AB = XY$, $AC = YZ$, $\angle B = \angle Z$.

14. $AB = XY$, $BC = XZ$, $\angle A = \angle Y$.

15. $AB = XZ$, $BC = YZ$, $\angle B = \angle Z$.

16. $AB = XY$, $BC = XZ$, $CA = YZ$.

17. $\angle A = \angle Z$, $AB = YZ$, $\angle C = \angle X$.

18. $\angle A = \angle Z$, $\angle B = \angle X$, $\angle C = \angle Y$.

(Other Examples on Congruent Triangles will be found on pages 72, 73, 79, 80.)

DRAWING OF TRIANGLES, QUADRILATERALS, ETC.

EXAMPLES 17.

(In every case draw a small freehand sketch and insert in it the magnitudes given in the question.)

Draw a triangle ABC with the given dimensions; measure all the other sides and angles.

1. $a = 1.9''$, $b = 1.3''$, $C = 68^\circ$.

2. $a = 2.4''$, $b = 1.7''$, $c = 1.4''$.

3. $a = 2.8''$, $B = 41^\circ$, $C = 50^\circ$.

4. $A = 38^\circ$, $C = 97^\circ$, $b = 4.2$ cm.

5. $A = 74^\circ$, $b = 4.7$ cm., $c = 3.1$ cm.

6. $a = 3.3''$, $b = 2.2''$, $c = 2.6''$.

7. $a = 4.1$ cm., $A = 51^\circ$, $B = 48^\circ$.

8. $b = 1.8''$, $c = 1.5''$, $B = 63^\circ$.

9. $c = 4.4$ cm., $b = 6.8$ cm., $B = 111^\circ$.

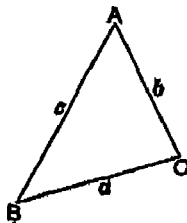


FIG. 83.

10. $a = 3.5''$, $b = 2.4''$, $c = 2.1''$.

11. $a = 2.15''$, $b = 1.65''$, $c = 2.15''$.

12. $a = 3.5$ cm., $b = 8.4$ cm., $c = 9.1$ cm.

13. A, B, C are three towns such that $AB = 5.2$ miles, $BC = 7.5$ miles, and $CA = 4.5$ miles. How far is C from the point midway between A and B?

14. AB is a straight road and AC is a railway, both running from the town A. $AB = 4.9$ miles, $AC = 6.2$ miles, and $\angle A = 72^\circ$. What is the perpendicular distance from B to the railway?

15. A, B, C are three church spires. B is 3.7 miles away from A in the direction N. 17° E.; C is 4.9 miles away from A in the direction S. 49° E. Find the distance from B to C and the bearing of B when viewed from C.

16. X and Y are two villages 15 miles apart. A man walks from X to Y along two straight roads (which meet at P) and he finds he has to walk 11 miles along the first and 12.5 miles along the second. Find the angle which the first road makes with the straight line XY.

Draw a quadrilateral ABCD with the dimensions given in Exx. 17-23.

17. $AB = 3.7$ cm., $BC = 1.9$ cm., $CD = 4.9$ cm., $DA = 3.9$ cm., $\angle D = 65^\circ$.

18. $DC = 2.5''$, $\angle D = 62^\circ$, $\angle C = 90^\circ$, $DA = 2.1''$, $BC = 1.7''$.

19. $AD = 5$ cm., $AB = 4.4$ cm., $\angle D = 65^\circ$, $\angle A = 93^\circ$, $\angle B = 125^\circ$.

20. $AB = 3.4$ cm., $AD = 5.1$ cm., $BC = 7.6$ cm., $DC = 6.2$ cm., diagonal $AC = 7.1$ cm.

21. $\angle D = \angle B = 100^\circ$, $AD = AB = 2.4''$, $\angle A = 71^\circ$.

22. $\angle B = 47^\circ$, $AB = 4.3$ cm., $BC = 6.5$ cm., $CD = 3.7$ cm., $AD = 3.4$ cm.

23. $AD = 1.7''$, $AB = 1.3''$, $\angle A = 117^\circ$, $\angle B = \angle D = 90^\circ$.

24. Draw a kite whose sides are $2.1''$ and $2.9''$, the angle between the two largest sides being 58° . Draw the diagonals. What do you think can be said about the way they cut? Try to see if the same fact seems to be true of any other kite you happen to draw. (See page 97 for explanation of the word *kite*.)

25. Draw a parallelogram whose sides are any lengths you please, using your set-square and ruler to draw the parallel lines. Measure the sides and angles. What seems to be true from your results? Test with two or three parallelograms of different shapes.

26. Draw a parallelogram by means of your ruler and set-square, and draw its diagonals. What do you notice about the way they cut each other? Test your statement with other parallelograms.

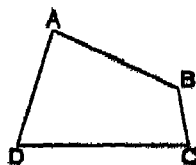


FIG. 84.

THE SIDES OF A RIGHT-ANGLED TRIANGLE.

EXAMPLES 18.

1. Draw a right-angled triangle ABC in which $\angle A = 90^\circ$, $AB = 3$ cm., and $AC = 4$ cm. Show by measurement that the hypotenuse (that is, the side opposite the right angle) is very nearly 5 cm. long.

[The average of all the results obtained in various attempts should be almost exactly 5 cm.]

Notice that $3^2 + 4^2 = 5^2$.

2. Take three other numbers a , b , c , such that $a^2 + b^2 = c^2$ (thus $5^2 + 12^2 = 13^2$). Draw a triangle with sides 5, 12, 13 units long (take any unit you please), and see whether the largest angle is a right angle.

3. Do example 2 again, using the numbers:

(i) 1.5, 2, 2.5; (ii) 9, 12, 15; (iii) 7, 24, 25.

(Show that each of these sets satisfies $a^2 + b^2 = c^2$.)

These examples suggest the truth of the fact that:

In a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides.

This fact is called Pythagoras' Theorem. Use it in the arithmetic of the following examples.

4. Calculate how far I am from home if I walk 8 miles W. and then 6 miles N. Verify by a figure drawn to scale.

5. Calculate the length of the diagonal of the floor of a hall which is 24 feet long and 32 feet wide. Verify by a drawing.

6. How high up a wall will a ladder 26 feet long reach if its foot is 10 feet away from the bottom of the wall? Work this out (i) by calculation; (ii) by drawing.

7. A ship sails 13 miles due N., 14 miles due E. and then 61 miles due S. How far is she now from her starting point?

8. The sill of a window is 60 feet above the ground, and the foot of a ladder is 11 feet away from the foot of the wall below the window. How long is the ladder if it just reaches to the window-sill?

SYMMETRICAL FIGURES.

EXAMPLES 19.

1. Take a piece of paper and fold it over so as to make a crease. Then cut out any pattern in the double paper from crease to crease. If the paper be now opened out the pattern will be repeated on the opposite sides of the crease (see Fig. 85).

Such a double figure, in which the two sides correspond exactly, is said to be *symmetrical* about the crease. The central line, which takes the place of the crease, is called the *axis of symmetry*.

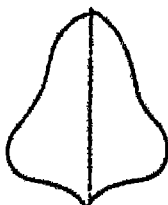


FIG. 85.

2. Take a piece of paper and fold it over so as to make a crease. Now fold it over again so that the crease folds over on itself. Then cut out any pattern in the four thicknesses of paper from one crease to the other. If the paper be now opened out, the pattern will be repeated four times over (see Fig. 86).

Such a four-fold figure, in which the four parts correspond exactly, is said to be a case of *double symmetry*. The full figure is symmetrical about each of the two axes of symmetry.



FIG. 86.

3. There is a third kind of symmetry, but this cannot be obtained by folding. Fig. 87 shows an instance of it. The curve is such that to every point P there is a corresponding point P' such that PP' always goes through a fixed point O, and PP' is always bisected at O. The curve is said to be *symmetrical about the point O*. O is called the *centre of symmetry*. [There is no axis of symmetry.] O must be the middle point of AB. Why?

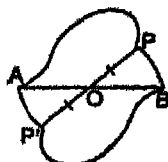


FIG. 87.

The figure could be drawn by taking any stencil pattern and drawing the part APB, and then, reversing the pattern and placing it below AB, by drawing the lower half AP'B.

Sometimes there is no definite break at A and B in the shape of the curve (see Fig. 88).

4. Describe what kinds of symmetry, if any, there are in the following printed letters:

A, B, D, E, H, I, L, M, N, O, S, T,
U, V, W, X, Y, Z.

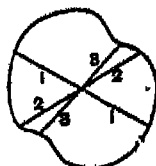


FIG. 88.

In each case show in a drawing the axis of symmetry or the centre of symmetry, if either exists.

(SHORTER)

E

5. Is symmetry about one axis a case of symmetry about a point? Is double symmetry about two axes a case of symmetry about a point? (Draw figures to illustrate the answers.)

6. Notice that a circle is a case of symmetry about a diameter; it is also a case of double symmetry about two diameters at right angles; it is also a case of symmetry about the centre. Draw figures to illustrate each of these statements.

7. Draw a figure to illustrate the fact that an isosceles triangle is a case of symmetry about a line from the vertex to the middle point of the base. Cut out an isosceles triangle in paper by folding and unfolding a piece of paper as in Ex. 1.

8. Show by means of a figure that a parallelogram is a case of symmetry about a point, the centre of symmetry being the point where the diagonals meet.

9. What kind of symmetry have the following figures?

- | | |
|---------------------|-------------------------------|
| (i) a rectangle; | (iv) an equilateral triangle; |
| (ii) a kite; | (v) a rhombus; |
| (iii) a semicircle; | (vi) a square. |

Draw figures to illustrate your answers.

[Definitions of some of the figures named above occur later in the book.]

10. Symmetry evidently involves repetition of a pattern (see Exx. 1, 2, 3 above). Can there be repetition of a pattern without there being any kind of symmetry? (Consider different patterns of wallpapers you meet with; they are, of course, full of repetition; do they all contain symmetry? do some of them?)

Sometimes the fact that a figure is symmetrical makes one sure of certain facts in the figure. For instance, in the case of an isosceles triangle ABC which is symmetrical about a line from the vertex A to the middle point D of the base BC , it must be true that $\angle B = \angle C$; also that $\angle BAD = \angle CAD$. It is evident that if a curve is symmetrical about an axis AB and the points P and P' correspond, then PP' must be perpendicular to AB , and PP' must be bisected by AB . Otherwise P and P' would not fall together when the figure is folded over about AB as crease.

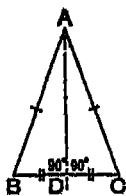


FIG. 89.

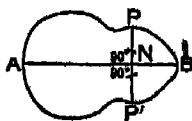


FIG. 90.

SIMILAR FIGURES.

It has been seen that if two triangles ABC, DEF have the angles A, B, C of one equal in order to the angles D, E, F of the other (thus $A = D$, $B = E$, $C = F$), the triangles have the same shape, but may be of different sizes. It therefore follows that the sides of the Δ ABC represent the sides of the Δ DEF on some scale or other.

$$\text{Thus } \frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}.$$

We may extend the idea to quadrilaterals.

ABCD is a quadrilateral field with sides and angles as shown. How can we draw a copy A'B'C'D' of it on paper?

Need we draw the new sides the same length as the sides they represent? And what about the size of the new angles?

Notice the way in which we treat the angles and the different way in which we treat the sides of the two figures. The angles must be made equal to the originals. But the sides need note. If we draw B'C' $\frac{1}{1000}$ of the size of BC, then we must make C'D' $\frac{1}{1000}$ of the size of CD, and so on.

ABCD and A'B'C'D' are called *similar figures*.

Notice that (i) the angles of one figure = the angles of the other, and (ii) the sides of one figure represent the sides of the other; the sides are said to be *proportional* and we

may write $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DA}{D'A'} = \frac{1000}{1}$, or whatever the fraction is which shows the relation in size.

Let us think again about copying a Δ ABC on a different scale so as to get two *similar triangles*.

I. We have had one way of doing so—making the angles of ABC equal to the angles of DEF (see p. 43).

Is there any other way?

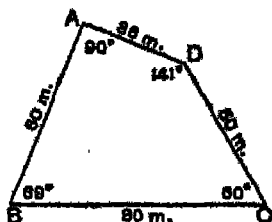


FIG. 91.

II. Consider the sides of the two triangles.

Make a triangle XYZ whose sides represent the sides of the triangle ABC on a scale of 10 yards to 1 inch. Measure the angles of XYZ.

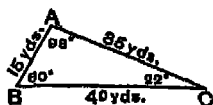


Fig. 92.

Make a triangle PQR whose sides represent the sides of the triangle ABC on a scale of 10 yards to 1 centimetre. Measure the angles of PQR.

What do you notice about the angles of $\triangle XYZ$ and of $\triangle PQR$?
Take another example:

Draw any triangle you please, and then draw another triangle whose sides are exactly half the sides of the first triangle. Measure the angles of the two triangles. What do you notice? Evidently

If two triangles have their sides proportional, their corresponding angles are equal.

III. Suppose we made a triangle XYZ in which $\angle X = \angle A$ and the sides round the angle X represent the sides round the angle A on a scale of 10 yds. to 1 inch. Measure the other angles. What do you notice? Evidently

If two triangles have one angle of one triangle equal to one angle of the other and the sides about those angles proportional, the triangles are similar.

EXAMPLES 20.

1. Are the following figures similar?

- | | |
|---|-------------------------|
| (i) A pair of equilateral triangles; | (iv) Any two squares; |
| (ii) A pair of isosceles triangles; | (v) Any two rectangles; |
| (iii) A pair of right-angled triangles; | (vi) Two circles. |

2. Show by drawing figures that two quadrilaterals can have the angles of one equal to the angles of the other without being similar.

3. Show by drawing figures that two quadrilaterals can have the sides of one proportional to, say one half of, the sides of the other without being similar.

4. Copy the quadrilateral ABCD on a scale of $\frac{1}{2}$, the angles being kept the same size. Do this:

- (i) Using sides as much as possible;
- (ii) Using angles as much as possible.

In each case show, by measuring, that:

- (i) All the angles of the new figure = in turn all the angles of ABCD;
- (ii) All the sides of the new figure are halves of the corresponding sides of ABCD.

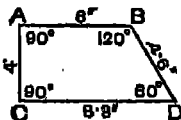


Fig. 93.

STAGE III.
THEORETICAL GEOMETRY

ABBREVIATIONS

The following abbreviations are in common use in Geometry; they are introduced gradually in this book, so that the pupil may get used to them one or two at a time:

adj.	adjacent.	$\nless$	is not less than.
alt.	alternate.	opp.	opposite.
$\angle$	angle; $\angle$ s angles.	$\parallel$	{ parallel (<i>adj.</i>). is (or are) parallel (to).
$\therefore$	because.	par ^l	parallel.
$\odot$	circle.	par ^m	parallelogram.
$\odot$ nce	circumference.	$\perp$	{ perpendicular (<i>adj.</i>). is (or are) perpendicular (to).
Cor.	corollary.	perp ^r	perpendicular.
corresp.	corresponding.	quad ^l	quadrilateral.
diam.	diameter.	rect.	rectangle.
ext.	exterior.	rt. $\angle$ (s)	right angle(s).
int.	interior.	sq.	square.
$=$	{ equal (<i>adj.</i>). is (or are) equal (to).	str.	straight.
$\equiv$	{ congruent (<i>adj.</i>). is (or are) congruent (to).	$\therefore$	therefore.
$\neq$	is not equal to.	trap ^m	trapezium.
$>$	is greater than.	Δ	triangle; Δ s triangles.
$\nless$	is not greater than.	vert.	vertically.
$<$	is less than.		

Q.E.D., placed at the end of each Theorem, stands for *Quod erat demonstrandum*, a phrase in Latin which may be translated thus: "Which was the thing that had to be proved."

Q.E.F., placed at the end of each Problem, stands for *Quod erat faciendum*, which means: "Which was the thing that had to be done."

NOTE.—The signs (*) and (†) are prefixed to the enunciations of certain propositions, as suggested in the I.A.A.M. report referred to above. According to this report "No formal proof of theorems marked (*) should be required for examination purposes. Of those marked (†) no formal demonstration should be attempted in the class-room; the truth of these theorems should be brought home to the pupil through experiment and intuition. The theorems marked (*) but not (†) are easily deducible logically from preceding theorems, and this may usefully be pointed out in the class-room."¹ Some teachers may, however, like to have formal "proofs" of these simple propositions, and so they are inserted in this book, but they can be ignored or used at the teacher's will.

SECTION I

MEANING OF TERMS USED IN GEOMETRY

The bookwork that follows in this book consists of a series of separate discussions or arguments. Each complete discussion is called a Proposition.

There are two kinds of Propositions—Theorems and Problems.

A Theorem is a Proposition in which one sets out to *prove some statement to be true*.

A Problem is a Proposition in which one sets out to *construct some figure*.

In a Problem, after stating the method of constructing the figure required, a proof is added to show that the construction is correct.

Examples of headings (or enunciations) of Propositions :

- (i) *The opposite angles of a parallelogram are equal.*
- (ii) *To bisect a given angle.*
- (iii) *To divide a straight line into any number of equal parts.*
- (iv) *The straight line joining the middle points of two sides of a triangle is parallel to the third side.*

Of these (i) and (iv) are Theorems; (ii) and (iii) are Problems.

There is also a different type of Problem, a practical one, in which the construction has to be done carefully, with ruler and compasses, certain lines, angles, etc., having to be made of given sizes. Thus a practical problem corresponding to the theoretical one stated above in (ii) would be:

Draw an angle of 63° and bisect it with ruler and compasses.

Here the construction lines would have to be shown, and it is generally expected that a short account of the method of construction should be added; but no proof need be given in a practical problem unless it is asked for.

The various parts of a Proposition are as follows :

(1) The general enunciation, which states in general terms the subject with which the problem deals (*i.e.* what has to be proved or constructed).

(2) The diagram, showing the particular figure to which the general enunciation is to be made to apply.

(3) The particular enunciation, which states (1) in particular terms, referring all lines, angles, etc., to the diagram in (2).

It consists of two parts :

(a) What is actually assumed or given.

(b) What is going to be proved (or constructed).

(4) The construction, which mentions whatever extra lines, etc., have to be drawn in the figure before the theorem can be proved (or the problem solved).

When there are no extra lines, etc., to be drawn in the figure, (4) is missing.

(5) The proof, which is the argument proper, proving the truth which the proposition set out to establish.

Theorems may be so simple that *no formal proof* is necessary. They are then called **Axioms**.

Problems may be so easy that *no formal construction* is necessary. They are then called **Postulates**.

(i) *To draw a circle with a given centre and a given radius.*

(ii) *Things that are double the same thing are equal.*

(i) is an example of a Postulate ; (ii) of an Axiom.

In the next set of Examples there are many instances of Axioms ; but as Axioms are self-evident truths (or common-sense truths), it should be possible to deduce the results of these easy arguments by common sense.

EXERCISES ON EASY ARGUMENTS.

EXAMPLES 21.

1. If $\angle A = \angle B$ and $\angle B = \angle C$, what is known about $\angle A$ and $\angle C$? [Obviously $\angle A$ and $\angle C$ are equal.]

2. XY and AB are two straight lines, both 2" long ; a portion is cut off from each, $\frac{1}{2}$ " long in each case. What can be said about the remainders ?

3. The line AB is twice as long as the line PQ , and the line CD is twice as long as PQ . What can you say about AB and CD ?

What follows from the statements in Examples 4 to 14?

4. $\angle A$ is three times $\angle B$, and $\angle C$ is three times $\angle B$.
5. Line $AB =$ line CD , and each line is continued an extra inch, AB being produced to X and CD being produced to Y .
6. ABC and XYZ are two straight lines; $AB = XY$ and $BC = YZ$.
7. $\angle A + \angle B = \angle A + \angle C$.
8. $\angle X + \angle Y + \angle Z = \angle A + \angle Y + \angle Z$.
9. $\angle A + \angle B = 90^\circ$; $\angle A + \angle X = 90^\circ$.
10. $\angle P + \angle Q = 180^\circ$; $\angle P + \angle R = 180^\circ$.
11. $\angle X$ is twice $\angle P$, $\angle Y$ is twice $\angle Q$, $\angle P = \angle Q$.
12. Line AB is half line CD ; line EF is half line CD .
13. Line AB is half line CD ; line CD is twice line EF .
14. $\angle A + \angle B + \angle C = 180^\circ$, $\angle A + \angle B + \angle D = 180^\circ$.
15. What can be said about:
 - (i) Two angles (P and Q), each of which is equal to another angle (R)?
 - (ii) Two lines, each of which is twice as long as another line?
16. If $\angle A = \angle P$ and $\angle P = 180^\circ$, what do you know about $\angle A$?
17. If $\angle X = \frac{1}{2} \angle Y$ and $\angle Y =$ two right angles, what do you know about $\angle X$?
18. If $\angle A = \frac{1}{2} \angle B$ and $\angle C = \frac{1}{2} \angle D$, and $\angle B + \angle D = 180^\circ$, what can you say about $\angle A + \angle C$?
19. If the sum of the angles X and Y is 30° , and $\angle P$ is twice $\angle X$ and $\angle Q$ is twice $\angle Y$, what do you know about the sum of the angles P and Q ?

What follows from the following statements?

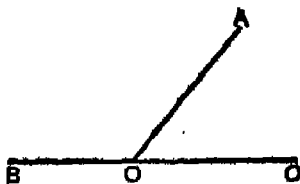
20. $\angle ABC + \angle XYZ = 180^\circ$ and $\angle ABC = 60^\circ$.
21. $A = B$ and B is greater than C .
22. $A + B = C + D$ and A is greater than C .
23. $\angle A$ is greater than $\angle B$, and $\angle B$ is greater than $\angle C$.
24. $\angle \alpha$ is the complement of $\angle \beta$ and $\angle \beta$ is the complement of $\angle \gamma$.
25. Each of the angles X and Y is the supplement of angle Z .
26. Each of two things (A and B) is equal to the same thing (C).
27. Each of two things is double of the same thing.
28. Each of two things is half of the same thing.
29. Each of two things is greater than the same thing.
30. $\angle X + 80^\circ = \angle Y$ and $\angle P + 80^\circ = \angle Z$; if $\angle P$ is twice $\angle X$, what relation is there between $\angle s Y$ and Z ?

*† THEOREM 1.

If a straight line stands on another straight line, the sum of the adjacent angles so formed is equal to two right angles.

Given a straight line AO standing on the straight line BOC .

Required to prove that the sum of the adjacent $\angle$ s BOA , AOC = two right $\angle$ s.



Proof. The two $\angle$ s BOA , AOC together
= the straight angle BOC ,
= two right angles.

Q.E.D.

Corollary I. If several straight lines stand on another line and meet it at the same point, the sum of the adjacent angles so formed is equal to two right angles.

Thus in Fig. 94 the angles α , β , γ together = a straight $\angle$ (i.e. two right $\angle$ s).



FIG. 94.

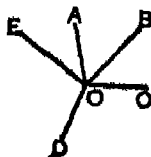


FIG. 95.

Corollary II. If any number of straight lines meet at a point, the sum of the adjacent angles so formed is equal to four right angles.

Thus in Fig. 95 the sum of the angles at O = four right $\angle$ s.

NOTE.—A Corollary is a Proposition which can easily be deduced from another Proposition (see above).

***† THEOREM 2.**

If the sum of two adjacent angles is equal to two right angles, the exterior arms of the angles are in the same straight line.

¹ Given two adjacent $\angle$ s BOA, AOC which are together equal to two right angles.

Required to prove that BO and OC are in a straight line.

Proof. The two $\angle$ s BOA, AOC together

= two right angles,

= a straight angle,

$\therefore$ BOO is a straight angle;

that is, BO and OC are in a straight line.

Q.E.D.

The proof of Theorem 2 uses the figure to Theorem 1.

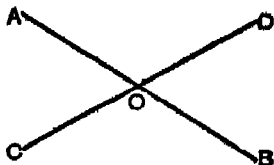
*** THEOREM 3.**

If two straight lines intersect, the vertically opposite angles are equal.

Given two straight lines AB and CD intersecting at O.

Required to prove that (i) $\angle$ AOC = $\angle$ BOD;

and (ii) $\angle$ AOD = $\angle$ COB.



Proof. $\angle$ AOC is the supplement of $\angle$ AOD, (*adj. $\angle$ s.*)

also $\angle$ BOD is the supplement of $\angle$ AOD; (*adj. $\angle$ s.*)

$\therefore \angle$ AOC = $\angle$ BOD.

Similarly it may be shown that $\angle$ AOD = $\angle$ COB.

Q.E.D.

MISCELLANEOUS EXERCISES ON ANGLES AT A POINT. COMPLEMENTARY AND SUPPLEMENTARY ANGLES.

EXAMPLES 22.

1. Find the complements of the following angles, drawing a figure to illustrate each case:

(i) 30° ; (ii) 15° ; (iii) 45° ; (iv) 70° ; (v) 85° .

2. Show, by drawing figures, whether the supplements of the following angles are acute or obtuse, and mark the number of degrees in each supplement:

(i) 40° ; (ii) 75° ; (iii) 145° ; (iv) 110° ; (v) $22\frac{1}{2}^\circ$.

3. If, in a $\triangle ABC$, $\angle B = \angle C$, and AB is produced to X and AC is produced to Y , show that $\angle XBC = \angle YCB$.

4. How many degrees are there in the supplement of a fifth of a right angle?

5. Find the number of degrees in an angle whose supplement is three times its complement.

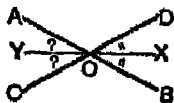


FIG. 96.

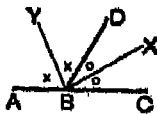


FIG. 97.

6. In Fig. 96 AB and CD are two lines meeting at O . If OX bisects $\angle BOD$, and XO is produced beyond O to Y , prove that OY is certain to bisect $\angle AOC$.

7. ABC is a straight line and BD is another drawn through B such that $\angle DBC = 56^\circ$ (see Fig. 97). If BX bisects $\angle DBC$ and BY bisects $\angle DBA$, calculate the size of each angle of the figure and find how many degrees there are in $\angle XBY$.

8. Calculate the size of $\angle XBY$ in Ex. 7 if $\angle DBC = 44^\circ$.

9. Prove, in the figure to Ex. 7, that $\angle XBY$ is *always* a right angle.

10. A man O walking in level country sees before him three objects A, B, C . He walks *towards* B , the middle and more distant object, and wishes to place himself in the line AO . He measures and finds $\angle AOB = 69^\circ$ and $\angle BOC = 113^\circ$. Is he now on AC , or must he go on or turn back?

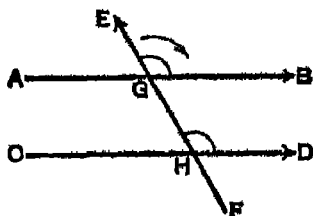
11. From O draw four lines in order, OA, OB, OC, OD ; if $\angle AOB = \angle COD$, and $\angle BOC = \angle DOA$, prove that (i) BOD is a straight line; (ii) AOC is also straight.

*† THEOREM 4.

When a straight line cuts two other straight lines and makes a pair of corresponding angles equal, the two straight lines are parallel.

Given a straight line EF cutting the straight lines AB and CD and making the $\angle EGB =$ the corresponding $\angle GHD$.

Required to prove that AB and CD are *par^l*.



Proof. If EF rotates clockwise about G until it lies in the direction of BA , it describes the $\angle EGB$,

and if EF also rotates clockwise about H until it lies in the direction of DC , it describes the $\angle GHD$;

but $\angle EGB = \angle GHD$, (*given*)

$\therefore$ EF must have rotated each time into the same direction,

$\therefore$ BA and DC are in the same direction,

that is, AB and CD are *parallel*.

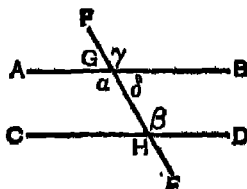
Q.E.D.

***THEOREMS 4a AND 4b.**

When a straight line cuts two other straight lines and makes

either (a) a pair of alternate angles equal,
or (b) a pair of interior angles on the same side of the cutting line supplementary,

then in either case the two straight lines are parallel.



4a. Given a straight line EF cutting two other straight lines AB and CD so that the alternate angles α and β are equal.

Required to prove that AB and CD are *par^l*.

Proof. $\alpha = \beta$, (given)
 and $\alpha = \gamma$, (vert. opp.)
 $\therefore \beta = \gamma$,
 and these are corresponding angles;
 $\therefore$ AB and CD are *parallel*.

Q.E.D.

4b. Given that (*in the same figure*) the two interior angles β and δ are supplementary.

Required to prove that AB and CD are *par^l*.

Proof. β is the supplement of δ , (given)
 and γ is also the supplement of δ ; (adj. $\angle$ s.)
 $\therefore \beta = \gamma$,
 and these are corresponding angles;
 $\therefore$ AB and CD are *parallel*.

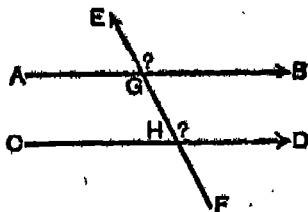
Q.E.D.

*† THEOREM 5.

If a straight line cuts two parallel straight lines, it makes corresponding angles equal.

Given a straight line EF cutting two parallel straight lines AB and CD.

Required to prove that $\angle EGB = \text{the corresponding } \angle GHD$.



Proof. If EF rotates clockwise about G until it lies along BA, it describes $\angle EGB$,

and if EF rotates clockwise about H until it lies along DC, it describes $\angle GHD$;

but, since AB and CD are parallel, EF rotates into the same direction in each case,

$\therefore$ EF rotates through equal $\angle$ s,

$\therefore \angle EGB = \angle GHD$.

Q.E.D.

MISCELLANEOUS EXERCISES ON PARALLELS.

EXAMPLES 23.

1. In the figure to Theorem 4 (p. 61), prove that, when the lines AB and CD are parallel,

(i) The $\angle$ s EGB, DHF together = two right angles;
and (ii) $\angle EGA = \angle DHF$.

2. In the figure used in Ex. 1 prove that the lines AB and CD are parallel if $\angle EGA + \angle CHF = 180^\circ$.

3. Draw a straight line and on it mark three points O, P, Q. At O, P, and Q draw straight lines (all on the same side of OPQ) making angles of 30° with the line OPQ. Show that in this way you get two sets of parallel lines. How do you know that the lines are parallel?

***THEOREMS 5a AND 5b.**

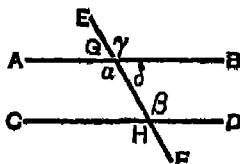
If a straight line cuts two parallel straight lines, it makes

- (a) alternate angles equal,
and (b) the interior angles on the same side of the cutting line supplementary.

Given a straight line EF cutting two par^l lines AB, CD.

Required to prove that

- (a) the alternate $\angle$ s α and β are equal,
(b) the interior $\angle$ s β and δ (on the same side of EF) are supplementary.



Proof. (a) Because EF meets the parallel lines AB and CD,

$$\begin{aligned} \therefore \beta &= \gamma, & (\text{corresp.}) \\ \text{but } \alpha &= \gamma, & (\text{vert. opp.}) \\ \therefore \alpha &= \beta. \end{aligned}$$

Q.E.D.

(b) Again,

$$\begin{aligned} \beta &= \gamma, & (\text{corresp.}) \\ \text{but } \delta &\text{ is the supplement of } \gamma, & (\text{adjacent}) \\ \therefore \delta &\text{ is the supplement of } \beta. \end{aligned}$$

Q.E.D.

EXAMPLES 23 (continued).

4. ABCD is a straight line, and BX and CY are drawn at right angles to AD through B and C. Show that BX and CY are parallel

- (i) when X and Y are on opposite sides of ABCD;
and (ii) when X and Y are on the same side of ABCD.

5. If a straight line is perpendicular to a line AB, it will be perpendicular to another line CD, if CD is drawn parallel to AB (see Fig. 98).

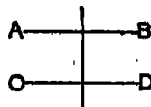


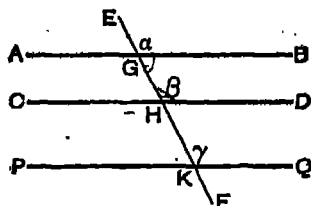
FIG. 98.

* THEOREM 6.

Straight lines which are parallel to the same straight line are parallel to one another.

Given two straight lines AB, CD which are both par^l to PQ.

Required to prove that AB and CD are par^l.



Construction. Draw a straight line EF cutting AB, CD, PQ at G, H, K.

Proof. EF cuts the par^l lines AB and PQ,
 $\therefore \alpha = \gamma$; (corresp.)
 again, EF cuts the par^l lines CD and PQ,
 $\therefore \beta = \gamma$; (corresp.)
 $\therefore \alpha = \beta$,
 $\therefore$ AB and CD are parallel. (corresp. $\angle s =$)

Alternative Proof. Since AB and PQ are par^l, AB is in the same direction as PQ.

Similarly CD is in the same direction as PQ.

$\therefore$ AB and CD are in the same direction,
 that is, AB and CD are parallel.

Q.E.D.

EXAMPLES 23 (continued).

6. Suppose that, in the figure to Theorem 4, GX bisects $\angle$ EGB and HY bisects $\angle$ GHD; prove that GX and HY are parallel.

7. Prove Theorem 6 when PQ is drawn between AB and CD.

8. In a $\triangle ABC$, $\angle B = \angle C$; BA is produced to X and AD is drawn parallel to BC. Prove that AD bisects $\angle XAC$ (see Fig. 99).



FIG. 99.

F

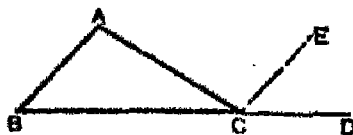
(SHORTER.)

THEOREM 7.

The sum of the angles of a triangle is equal to two right angles.

Given a $\triangle ABC$.

Required to prove that the sum of the $\angle$ s A, B, and C equals two right $\angle$ s.



Construction. Produce the side BC to D;

* and suppose CE to be the line through C which is parallel to AB.

Proof. $\because$ BA and CE are parallel and AC meets them,
 $\therefore \angle A = \text{alternate } \angle ACE$;

Again, $\because$ BA and CE are parallel and BCD meets them,
 $\therefore \angle B = \text{corresponding } \angle ECD$;
 $\therefore \angle A + \angle B = \text{whole } \angle ACD$.

Add $\angle ACB$ to each;
 $\therefore \angle A + \angle B + \angle ACB = \angle ACD + \angle ACB$.

But $\angle ACD + \angle ACB = \text{two right } \angle$ s; (adjacent)

$\therefore$ the sum of the three $\angle$ s A, B, C of the $\triangle ABC = \text{two right } \angle$ s.

Q.E.D.

Cor. 1. If one side of a triangle is produced, the exterior angle so formed is equal to the sum of the two interior opposite angles.

Thus the exterior $\angle ACD = \angle A + \angle B$.

This is proved in the first five lines of the above proof.

Cor. 2. If two triangles have two angles of the one equal to two angles of the other, each to each, then the third angles are also equal.

* See note on Hypothetical Constructions on page 75.

Cor. 3. *Every triangle has at least two acute angles.*

For if a triangle had only one acute angle, each of the other two angles would be 90° or more, and this is impossible, for the sum of the three angles would now be greater than 180° .

Cor. 4. *In a right-angled triangle—*

- (i) *The right angle is the greatest angle;*
- (ii) *The sum of the two remaining angles is equal to a right angle.*

EXERCISES ON THE ANGLES OF A TRIANGLE.

EXAMPLES 24.

1. If the three angles of a triangle are all equal, express the size of each: (i) as a fraction of a right angle; (ii) in degrees.

2. If two angles of a triangle are equal, prove that each is the complement of half the third angle.

3. Prove that the four angles of a quadrilateral are together equal to four right angles. (Draw a diagonal.)

4. ABCD is a quadrilateral and AC, BD cut in O. If $\angle DAC = 52^\circ$, $\angle BAC = 47^\circ$, $\angle ADB = 42^\circ$, $\angle ADC = 70^\circ$, $\angle ABC = 81^\circ$, calculate the size of all the other angles of the figure, and show that AD and BC are parallel.

5. ABC is a triangle in which $\angle B = \angle C$, and BA is produced to P. Prove that $\angle PAC$ is twice $\angle C$.

6. Prove that an isosceles Δ is equilateral if either the vertical angle or one of the angles at the base is 60° .

7. In a ΔABC , $\angle B = 60^\circ$ and $\angle C = 84^\circ$; BI and CI bisect $\angle$ s B and C. Calculate $\angle A$ and $\angle I$, and show that $\angle I = 90^\circ + \frac{1}{2} \angle A$.

8. If $\angle A$ of a ΔABC is 52° , and BI and CI bisect the base angles B and C, calculate $\angle BIC$.

9. Prove that, in Figure 101, the sum of the exterior angles ACD, BAE, CBF equals four right angles.

10. ABC is a triangle in which $\angle B = \angle C$; BA is produced to X and the exterior angle XAC is bisected by the line AY. Prove that AY and BC are parallel.

11. Prove that, if the base angles of a triangle are bisected (see Fig. 100), the angle (BIC) between the bisectors = a right angle + half angle A.

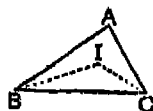


FIG. 100.

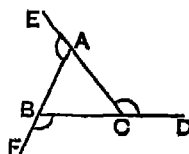


FIG. 101.

Example.—Draw polygons of 5, 6, 7 sides in turn, and, by dividing them into triangles, find the number of right angles in the sum of their angles. Is there any connection between these numbers and twice the number of sides of each figure? Can you deduce the number of right angles in a polygon of (i) 10; (ii) n sides?

THEOREM 8.

In a polygon of n sides, the sum of the interior angles is equal to $2n - 4$ right angles.

Given a polygon of n sides represented by ABCDE.

Required to prove that the sum of its interior angles = $(2n - 4)$ right $\angle$ s.

Construction. Join A to all the other angles except B and E.

Proof. The figure is now divided into $(n - 2)$ Δ s,

and since the three $\angle$ s of each Δ
= two right $\angle$ s,

$\therefore$ all the $\angle$ s of all the Δ s
= twice $(n - 2)$ right $\angle$ s,
= $(2n - 4)$ right $\angle$ s;

but all the $\angle$ s of the Δ s make up exactly the interior $\angle$ s of the figure ABCDE,

$\therefore$ the sum of the interior angles of the figure
= $(2n - 4)$ right $\angle$ s.

Q.E.D.

Corollary. If the sides of a convex polygon are produced in order, the sum of the exterior angles so formed is equal to four right angles.

[For the sum of the two $\angle$ s at any corner (one interior and one exterior) = 2 right $\angle$ s;

$\therefore$ all the interior $\angle$ s + all the exterior $\angle$ s = $2n$ right $\angle$ s;

but all the interior $\angle$ s = $2n - 4$ right $\angle$ s, (proved above)

$\therefore$ all the exterior $\angle$ s
= $2n - (2n - 4)$ right $\angle$ s
= 4 right $\angle$ s.]

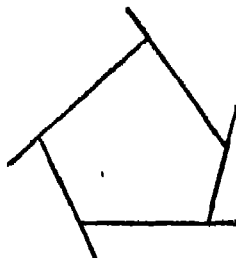
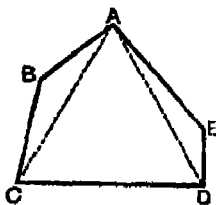


FIG. 102.

EXERCISES ON RECTILINEAR FIGURES.

EXAMPLES 25.

1. Express in (a) right angles, (b) degrees, the sum of the angles in:

- | | |
|----------------------|------------------|
| (i) a quadrilateral; | (iii) a hexagon; |
| (ii) a pentagon; | (iv) an octagon. |

2. If three of the angles of a quadrilateral are 95° , 73° , 68° , find the fourth angle.

3. Four angles of a pentagon are 75° , 86° , 117° , 130° . What is the fifth angle?

4. Find the number of degrees in each exterior angle of a regular (i) hexagon; (ii) pentagon; (iii) decagon; (iv) polygon of 36 sides.

5. Prove that each exterior angle of a regular polygon of n sides is $\frac{360^\circ}{n}$.

6. Find the size of each interior angle of a regular: (i) pentagon; (ii) decagon; (iii) octagon; (iv) polygon of 60 sides.

7. Find the number of corners and sides in a polygon if each exterior angle is 20° .

8. Answer No. 7 if each interior angle is 150° .

9. Find the number of sides in a polygon if each exterior angle is (i) 40° ; (ii) 36° ; (iii) 10° ; (iv) 5° ; (v) x° .

10. Find the number of sides in a polygon if each interior angle is (i) 108° ; (ii) 120° ; (iii) 174° ; (iv) 170° ; (v) 165° .

11. Five of the angles of a hexagon are 150° , 100° , 73° , 145° , 95° . Calculate the sixth angle:

- (i) by means of Theorem 8, using interior angles;
 (ii) by means of the Corollary to Theorem 8, using exterior angles.

12. A pentagon has angles $2A$, $3A$, $4A$, $5A$, $6A$. Find the size of each angle.

13. The interior angles A , B , C , D of a pentagon $ABCDE$ are 122° , 86° , 154° , 62° . Calculate the acute $\angle$ between the bisectors of the angles B and E .

14. $ABCDEF$ is a regular hexagon; prove that AB and ED are parallel.

15. The floor of a school-hall is paved with equal tiles shaped like a regular polygon. What particular type of polygon could be used for the tiles to fit together exactly at any point? [Give as many answers as possible and draw sketches to illustrate each answer.]

*† THEOREM 9.

If two triangles have two sides of the one equal to two sides of the other, each to each, and also the angles included by those sides equal, the triangles are congruent.

Given two Δ s ABC, DEF, in which $AB = DE$, $AC = DF$, and the included $\angle$ s A and D are equal.

Required to prove that Δ s ABC, DEF are congruent.

Proof. If the Δ ABC be drawn to have the sides AB, AC and the $\angle$ A of the exact sizes given,

AB can be drawn in only one way;

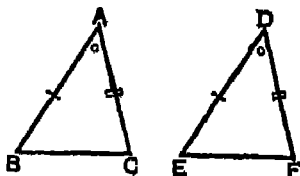
$\angle$ A can then¹ be drawn in only one way;

and AC can then be drawn in only one way;

$\therefore$ the Δ ABC can be drawn in only one way;

hence Δ DEF, which agrees with Δ ABC as regards the two sides and included $\angle$, must be an exact copy of Δ ABC,

that is, the Δ s ABC, DEF are congruent. Q.E.D.



Alternative Proof. Pick up Δ ABC and put it on Δ DEF, making AB coincide with the equal side DE;

then $\because \angle A = \angle D$, AC will fall along DF;

and $\because AC = DF$, C will fall on F.

Then, since B coincides with E and C with F,

BC must coincide with EF,

$\therefore \Delta$ ABC coincides with Δ DEF,

that is, the Δ s ABC, DEF are congruent. Q.E.D.

N.B.—When this Congruence is quoted in subsequent work, it will be referred to under the heading “Two sides and included angle.” Be careful and see that the angle taken in each triangle is always included between the given sides in that triangle.

¹ It evidently does not matter, as far as the size and shape of the Δ ABC are concerned, on which side of AB the angle A is drawn.

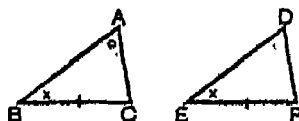
*† THEOREM 10.

If two triangles have two angles of the one equal to two angles of the other, each to each, and also a side of one equal to the corresponding side of the other, the triangles are congruent.

Given two Δ s ABC, DEF, in which :

$\angle A = \angle D$, $\angle B = \angle E$, and $BC = EF$.

Required to prove that Δ s ABC, DEF are congruent.



Proof. Since $\angle A = \angle D$ and $\angle B = \angle E$,

$\therefore \angle C = \angle F$. (3 $\angle$ s of a $\Delta = 2\text{rt. } \angle$ s.)

If the Δ ABC be drawn to have the side BC and the $\angle$ s B and C of the exact sizes given,

BC can be drawn in only one way ;

$\angle$ B can then be drawn in only one way ;

and $\angle$ C can then be drawn in only one way ;

$\therefore$ the Δ ABC can be drawn in only one way ;

hence Δ DEF, which agrees with Δ ABC as regards the two $\angle$ s and the given side, must be an exact copy of Δ ABC, that is, the Δ s ABC, DEF are congruent. Q.E.D.

Alternative Proof. As in above proof $\angle C = \angle F$.

Pick up Δ ABC and put it on Δ DEF, making BC coincide with the equal side EF ;

then $\because \angle B = \angle E$, BA will fall along ED,

and $\because \angle C = \angle F$, CA will fall along FD.

Then the point A, where BA and CA meet, will fall on the point D, where ED and FD meet ;

$\therefore \Delta$ ABC coincides with Δ DEF,

that is, the Δ s ABC, DEF are congruent. Q.E.D.

N.B.—When this Congruence is quoted in subsequent work, it will be referred to under the heading "Two angles and a side." The sides taken must correspond, that is, they must be opposite equal angles.

EXERCISES ON CONGRUENT TRIANGLES.

EXAMPLES 28.

THEOREM 9. ("TWO SIDES AND INCLUDED ANGLE.")

1. Two unequal lines, AD , BC cross in E , and E is the middle point of each line. If AB and CD are joined, prove that $AB = CD$. (Prove $\triangle AEB$, $\triangle CED$ congruent.)

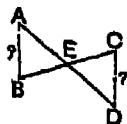


FIG. 103.

2. Prove that the straight line AD which bisects the vertical angle of a $\triangle ABC$ in which $AB = AC$ makes equal $\angle$ s ADB , ADC with the base BC . Show also that AD is perpendicular to BC .

3. ABC is a $\triangle$ in which $AB = AC$, and AD bisects the $\angle BAC$. In AD take any point K , and join KB and KC . Prove that $\triangle KBA$ and $\triangle KCA$ are congruent and that $KB = KC$.

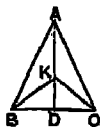


FIG. 104.

4. $ABCD$ is a four-sided figure in which $AB = CD$ and $\angle ABC = \angle BCD$. If O is the middle point of BC , prove that $AO = DO$.

THEOREM 10. ("TWO ANGLES AND A SIDE.")

5. BAC is any angle and AK bisects it; P is any point on AK , and PX is drawn perpendicular to AB and PY perpendicular to AC . Show that $PX = PY$ (see Fig. 105).

6. ABC is a triangle in which $\angle B = \angle C$; AD bisects $\angle BAC$ and meets BC in D . Show that the $\triangle$ s ABD , ACD are congruent and that $BD = DC$. What other lines and angles in the figure must be equal?



FIG. 105.

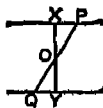


FIG. 106.

7. PQ is any straight line drawn to meet two parallel lines in P and Q , and O is the middle point of PQ . If XY be another straight line through O meeting the parallel lines in X and Y , show that $OX = OY$ (see Fig. 106).

8. In the figure to Theorem 10 (p. 71) AX is drawn perpendicular to BC and DY perpendicular to EF ; prove that $AX = DY$.

THEOREMS 9 AND 10.

9. PQRS is a four-sided figure. If $PQ = PS$ and the line PR bisects $\angle QPS$, prove that $QR = RS$.

10. ABC is a triangle in which $AB = AC$; BD is drawn perpendicular to AC and CE perpendicular to AB. Prove that the Δ s ACE, ABD are congruent and so show that $BD = CE$ (see Fig. 107).

11. In a triangle the two base angles are equal and the base is bisected at O; prove that the perpendiculars drawn from O to the other two sides of the triangle are equal.

12. A circle is drawn with centre O, and four radii OA, OB, OC, OD are drawn making $\angle AOB$ equal to $\angle COD$; join AB and CD, and prove that these lines are equal (see Fig. 108).

13. In No. 12 join AC and BD. Prove that:

(i) $\angle AOC = \angle BOD$, and

(ii) the straight lines AC, BD are equal.

14. If one diagonal of a four-sided figure bisects each of the angles it joins, prove that the figure has two pairs of equal sides.

15. Show that the breadth AB of a river with a straight bank through A as in Fig. 109 can be found thus:

Mark off from A at right angles to AB a length AC and then beyond C and in the same direction an equal length CD. Place a peg in the ground at C. From D walk to E at right angles to AD until you find that when you are at E the peg C is exactly between you and the point B. Then DE is equal to the breadth of the river.



Fig. 107.

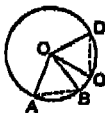


Fig. 108.

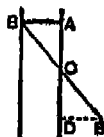


Fig. 109.

16. Draw an isosceles triangle ABC in which $AB = AC$; from AB and AC respectively cut off equal lengths AP and AQ; join BQ and CP. Prove that $BQ = CP$. (Take Δ s BQA, CPA.)

17. AB and AC, the equal sides of an isosceles triangle ABC, are produced beyond B and C to F and G so that $BF = CG$. Join BG and CF and prove that $BG = CF$. (Take Δ s ACF, ABG.)

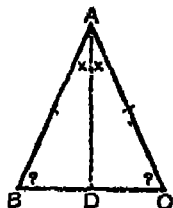
18. AOB, COD are two isosceles triangles with a common vertex O, and $\angle AOB = \angle COD$. Prove that, if AC and BD are joined, $AC = BD$.

THEOREM 11.

If two sides of a triangle are equal, the angles opposite those sides are equal.

Given a $\triangle ABC$ in which $AB = AC$.

Required to prove that $\angle B = \angle C$.



Construction. Suppose the line AD bisects $\angle BAC$ and meets BC in D .

Proof.

In the $\triangle s$ BAD , CAD ,

$AB = AC$,

(given)

$\left\{ \begin{array}{l} AD \text{ is common to both } \triangle s, \\ \text{and } \angle BAD = \angle CAD, \end{array} \right.$

(constr.)

$\therefore$ the $\triangle s$ are congruent, (two sides and incl. $\angle$)

so that $\angle B = \angle C$.

Q.E.D.

Cor. 1. If a triangle is equilateral, its angles are all equal.

Cor. 2. The angle in a semicircle is a right angle.

For if BAC be the angle in the semicircle on the diameter BOC , of which O is the centre, we have;

In $\triangle OAB$, $OA = OB$, (radii)

$\therefore \angle OAB = \angle B$,

and in $\triangle OAC$, $OA = OC$, (radii)

$\therefore \angle OAC = \angle C$.

$\therefore$ whole $\angle BAC = \angle B + \angle C$,

$\therefore \angle BAC =$ half of the three $\angle s$ of the $\triangle ABC$,

i.e. $\angle BAC =$ a right angle.

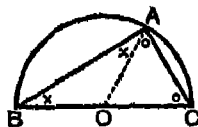


FIG. 110.

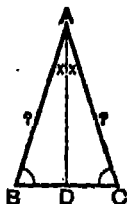
Q.E.D.

THEOREM 12.

If two angles of a triangle are equal, the sides opposite those angles are equal.

Given a $\triangle ABC$ in which $\angle B = \angle C$.

Required to prove that $AB = AC$.



Construction. Suppose the line AD bisects $\angle BAC$ and meets BC in D .

Proof.

In the $\triangle s$ BAD , CAD ,

$$\begin{cases} \angle B = \angle C, & \text{(given)} \\ \angle BAD = \angle CAD, & \text{(constr.)} \\ \text{and } AD \text{ is common,} \end{cases}$$

$\therefore$ the $\triangle s$ are congruent, (*two $\angle s$ and a side*)
so that $AB = AC$.

Q.E.D.

Cor. If a triangle is equiangular, it is also equilateral.

HYPOTHETICAL CONSTRUCTIONS.

In Theorems 7, 11 and 12 the construction is said to be a hypothetical one, as the method of bisecting an angle has not yet been reached. (See Problem 1, page 81.)

The argument in these Theorems, however, is still sound, since we can argue about a line which we know to exist, even though we do not know its exact position. We can even argue soundly about things that do not exist; e.g. if Jack climbed up the magic beanstalk 100 feet and slipped down 20 feet, we *know* that he is now 80 feet from the ground.

The argument does not even depend upon our having an accurate picture of the beanstalk in front of us.

EXERCISES ON ISOSCELES TRIANGLES.

EXAMPLES 27.

THEOREMS 11 AND 12.

1. If the vertical angle of an isosceles triangle is 40° , what is the size of each of the base angles?

2. What is the size of each of the smaller angles of a right-angled isosceles triangle?

3. ABC is a Δ in which $AB = AC$, and BC is produced both ways to X (beyond B) and to Y (beyond C). Show that $\angle ABX = \angle ACY$.

4. If the perpendicular to the base from the vertex of a Δ bisects the vertical $\angle$ of the Δ , prove that the Δ is isosceles.

5. Prove Theorem 12 by supposing that AD is drawn perpendicular to BC . See that you cannot prove Theorem 11 by the same method.

6. In a ΔABC , $AB = AC$, and $\angle B$ and $\angle C$ are bisected by BI and CI , meeting at I . Prove that $IB = IC$.

7. In a ΔABC , $AB = AC$; Y is the middle point of AC and Z of AB ; join BY and CZ . Prove that $BY = CZ$.

8. $DBCE$ is the base BC of an isosceles ΔABC produced at both ends so that $BD = CE$; join AD and AE . Prove that $AD = AE$.

9. In ΔABC , $AB = AC$ and $\angle B = 36^\circ$. D is a point in BC such that $BD = BA$. If AD is joined, prove that $AD = DC$.

10. ABC is an isosceles Δ , and BC , its base, is produced to X . BI bisects $\angle B$ and CI bisects $\angle ACB$. Prove that the exterior $\angle ACX = \angle BIC$.

11. ABC is a Δ in which $AB = AC$ and P is any point in BC (except the middle point); in the Δ s ABP , ACP we have $AB = AC$, AP is common, and $\angle ABP = \angle ACP$. Does this prove that the Δ s are congruent? (Give reasons for your answer.)

12. Prove that the Δ cut off from an isosceles Δ by a line drawn parallel to the base will be itself isosceles.

13. In a ΔABC , BK , which bisects $\angle ABC$, meets AC in K . Draw KM parallel to BC , to meet AB in M . Prove that $MK = MB$.

14. In a ΔABC the bisectors of $\angle B$ and $\angle C$ meet in I ; through I a line is drawn parallel to the base BC , meeting AB in X and AC in Y . Prove that $XY = BX + CY$.

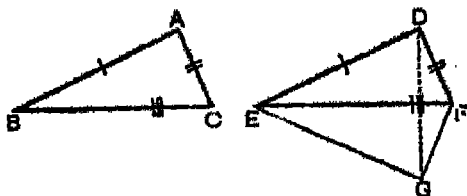
*† THEOREM 13.

If two triangles have the three sides of the one equal to the three sides of the other, each to each, the triangles are congruent.

Given two Δ s ABC, DEF in which

$$AB = DE, AC = DF, BC = EF.$$

Required to prove that *the Δ s ABC, DEF are congruent.*



Proof. Let BC be a side of Δ ABC which is longer than the other sides (or at least as long as each of the other sides).

Apply Δ ABC to DEF so that the equal sides BC, EF coincide, and A falls at G, on the side of EF away from D; join DG.

Then in the Δ EDG, $\because ED = EG$, (given)
 $\therefore \angle EDG = \angle EGD$; (opp. = sides)

Also in the Δ FDG, $\because FD = FG$, (given)
 $\therefore \angle FDG = \angle FGD$; (opp. = sides)

$\therefore$ by adding, we get, the whole $\angle D =$ the whole $\angle G$,
 which = the $\angle A$. (constr.)

Then in the Δ s ABC, DEF,

$$\left\{ \begin{array}{l} AB = DE, \\ AC = DF, \\ \text{and } \angle BAC = \angle EDF, \end{array} \right. \begin{array}{l} \text{(given)} \\ \text{(given)} \\ \text{(proved)} \end{array}$$

$\therefore$ the Δ s ABC, DEF are congruent. (two sides and incl. $\angle$)

Q.E.D.

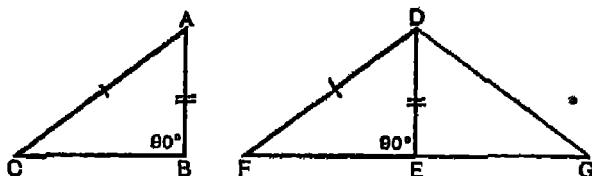
N.B.—When this Congruence is quoted in subsequent work, it will be referred to under the heading “Three sides.”

THEOREM 14.

If in two right-angled triangles the hypotenuses are equal, and one side of one triangle equals one side of the other, the triangles are congruent.

Given two right-angled Δ s ABC, DEF in which B and E are right angles, also hypotenuses AC, DF are equal, and side AB = side DE.

Required to prove that the Δ s ABC, DEF are congruent.



Proof. Apply Δ ABC to Δ DEF so that the equal sides AB, DE coincide, and C falls at G, on the side of DE away from F.

Then $\because$ the angles at E are right angles, (given)
 $\therefore$ GEF is a straight line. (adj. $\angle$ s supp.)

Hence DGF is a Δ in which DG = DF,
 $\therefore \angle F = \angle G$, (opp. equal sides)
 which = $\angle C$. (constr.)

Then in the Δ s ABC, DEF,
 $\left\{ \begin{array}{l} \angle C = \angle F, \\ \angle ABC = \angle DEF, \\ \text{and } AB = DE, \end{array} \right.$ (proved)
 (right $\angle$ s)
 (given)

$\therefore$ the Δ s ABC, DEF are congruent. (two $\angle$ s and a side)

Q.E.D.

N.B.—When this Congruence is used in subsequent work, it will be referred to under the heading "Right angle, hypotenuse, one side," or, more briefly, "Rt. $\angle$, hyp., 1 side."

FURTHER EXERCISES ON CONGRUENT TRIANGLES.

EXAMPLES 28.

THEOREM 13. ("THREE SIDES.")

1. ABC is an isosceles Δ in which $AB = AC$; on the other side of the base BC an equilateral ΔBCX is drawn. Join AX and prove that AX bisects $\angle BAC$.

2. PQ is a chord of a circle whose centre is O , and A is the middle point of PQ . Join OA and prove that:

(i) $\angle OAP = \angle OAQ$;

and

(ii) OA is perpendicular to PQ .

3. AB and CD are equal chords of a circle whose centre is O . Prove that $\angle AOB = \angle COD$.

4. The line from the vertex of an isosceles Δ which bisects the base is perpendicular to the base.

5. If the opposite sides of a quadrilateral are equal, prove that the opposite angles are also equal. (Draw a diagonal.)

THEOREM 14. ("RIGHT ANGLE, HYPOTENUSE, ONE SIDE.")

6. AB is a chord of a circle whose centre is O , and OX is drawn perpendicular to AB . Prove that X bisects AB .

7. Prove that the perpendicular from the vertex of an isosceles triangle on to the base bisects the base.

8. If a point P is such that the perpendiculars PM , PN drawn to two lines intersecting at O are equal, prove that PO bisects the angle between the lines.

9. If the perpendiculars drawn to the two sides of a triangle from the middle point of the base are equal, show that the triangle is isosceles.

10. If the perpendiculars from the ends of the base of a triangle to the opposite sides are equal, the triangle is isosceles.

MISCELLANEOUS EXERCISES ON THE CONGRUENCES.

EXAMPLES 29.

1. $\triangle ABC$ is a $\triangle$ in which $AB = AC$ and $AD \perp BC$. Prove that $\triangle ABD \equiv \triangle ACD$ by two different congruences.

2. $\triangle ABC$ is an isosceles $\triangle$ in which $AB = AC$ and D bisects AB and E bisects AC . Prove that $BE = CD$ by taking

- (i) $\triangle s ADC, AEB$; (ii) $\triangle s BDC, BEC$.

3. AB and CD are equal and parallel straight lines; join AC and BD so that they do not intersect. Prove that AC and BD are

- (i) equal; (ii) parallel.

4. In No. 3 AD and BC are joined and they cut in O . Prove that AD and BC are bisected at O .

5. $\triangle ABC$ is a $\triangle$ in which $AB = AC$. At B and C lines BX and CY are drawn $\perp BC$, and BX meets CA produced in X , and CY meets BA produced in Y . Prove that $BX = CY$ and $AX = AY$.

6. $\triangle ABC$ is an equilateral $\triangle$, and equilateral $\triangle s XAB, YAC$ are drawn on AB and AC as bases outside the original $\triangle$. If XC and YB are joined and cut in Z , prove that $XC = YB$ and that $BZ = CZ$.

7. In Ex. 6 prove that $XC = YB$ whatever be the shape of the triangle ABC .

8. AB and CD are two chords of a circle cutting each other at O and they make equal angles with the line joining O to K , the centre of the circle. Prove that:

- (i) the perpⁿ KP, KQ from K to AB, CD are equal;
 (ii) $AP = DQ$;
 (iii) the parts of the line $AB =$ the parts of the line CD .

9. $ABCDEF$ is a regular hexagon; join AC and BF , meeting at O . Prove that (i) $AC = BF$, and then that (ii) $OC = OF$.

10. $\triangle ABC$ is an isosceles $\triangle$ in which $AB = AC$. Produce AB to X and AC to Y ; making $BX = CY$; join CX and BY . Prove that:

- (i) $\triangle s ABY, ACX \equiv$, and then, without using Theorem 11, that
 (ii) $\triangle s BCX, CBY \equiv$.

Show now that $\angle ABC = \angle ACB$. [Euclid's proof of Theorem 11.] This was the Proposition in Euclid that was always referred to as *pons asinorum*, or *donkey's bridge*; pupils often found its difficulties hard to get over. Notice how much simpler the proof of Theorem 11 becomes when the hypothetical construction of page 74 is used.

PROBLEM 1.

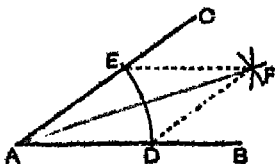
To bisect a given angle.

Given an angle BAC .

Required to *bisect* $\angle BAC$.

Construction. With centre A and any radius draw an arc cutting AB in D and AC in E .

With centres D and E and with any suitable radius draw arcs cutting at F .



Join AF .

Then *shall* AF *bisect* $\angle BAC$.

Proof.

Join DF , EF .

In the Δs DAF , EAF ,

$\begin{cases} AD = AE, & (\text{constr.}) \\ AF \text{ is common,} \\ \text{and } DF = EF, & (\text{constr.}) \end{cases}$

$\therefore$ the Δs are congruent, (3 sides)

so that $\angle DAF = \angle EAF$,
that is, AF *bisects* $\angle BAC$.

Q.E.F.

EXAMPLES 30 (mostly Practical).

PROBLEM 1.

1. Draw, with a protractor, angles of (i) 73° , (ii) 51° , (iii) 115° , and bisect each angle, using your compasses. Measure the two parts of each angle with your protractor.

2. How would you use your compasses to find out whether a particular line OP drawn on paper does bisect a given $\angle XOY$?

3. Draw a ΔABC given that $BC = 2''$, $\angle B = 70^\circ$, $\angle C = 60^\circ$, and bisect the three angles. [In an accurate figure the three bisectors go through one point.]

4. Prove that the construction in Problem 1 will still hold if the arcs at F are drawn on the other side of DE . There are two cases:

(i) When F is in the area between DE and A ;

(ii) When F is beyond A . In this case show that when FA is produced beyond A it will bisect $\angle A$.

G

(SHORTEN.)

PROBLEM 2.

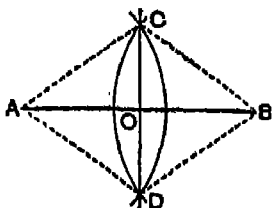
To bisect a given straight line.

Given a straight line AB.

Required to bisect AB.

Construction. With centre A and any radius greater than half AB describe an arc.

With centre B and the same radius describe an arc cutting the former arc in C and D.



Join CD, cutting AB in O.

Then *shall* O bisect AB.

Proof.

Join AC, CB, AD, DB.

In the Δ s ACD, BCD,

$$\begin{cases} AC = BC, & (\text{constr.}) \\ AD = BD, & (\text{constr.}) \\ CD \text{ is common,} & \end{cases}$$

$\therefore$ the Δ s are congruent, 3 sides

so that $\angle ACD = \angle BCD$.

Again,

in the Δ s ACO, BCO,

$$\begin{cases} AC = BC, & (\text{constr.}) \\ CO \text{ is common,} \\ \angle ACO = \angle BCO, & (\text{proved}) \end{cases}$$

$\therefore$ the Δ s are congruent, (2 sides & incl. $\angle$)

so that $AO = BO$,

that is, O bisects AB.

Q.E.F.

In the above figure CD is a line bisecting AB at right angles. This problem could therefore be enunciated thus:

To draw the perpendicular bisector of a given finite straight line.

N.B.—The *proof* given above can be used to prove the following important truth:

The diagonals of a kite cut at right angles, and the diagonal which joins the points where the equal sides meet (i.e. C and D) bisects the other diagonal. (See definition on page 97.)

Exercises on Problem 2 will be found on page 85.

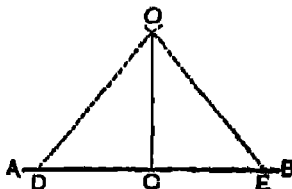
PROBLEM 8A.

To construct a perpendicular to a given straight line from a given point in the line.

Given a straight line AB and a point C in AB.
Required to draw from C a perpendicular to AB.

Method I. With centre C and any radius draw arcs cutting CA in D and CB in E.

With D and E as centres, and any length greater than DC as radius, draw two arcs meeting in O.



Join OC.

Then shall CO be perpendicular to AB.

Outline of Proof. For, joining OD and OE,
the Δ s OCD, OCE can be proved congruent, (3 sides)
 $\therefore$ the adjacent angles OCD, OCE are equal,
 $\therefore$ they are right angles, (= adj. $\angle$ s)
and CO is perp^r to AB.

Q.E.F.

Method II. With centre C and any radius draw an arc DEF cutting AB in D; with D as centre and the same radius draw an arc cutting arc DEF in E; with E as centre and the same radius draw an arc cutting arc DEF in F.

Join CE and CF.

Bisect $\angle$ ECF by the line CO.

Then shall CO be perp^r to AB.

Outline of Proof. For, joining DE and EF,

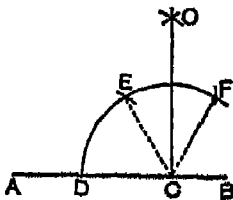
$\angle$ DCE = 60° . (equilateral Δ)

Similarly $\angle$ ECF = 60° .

$\therefore \angle$ ECO = half of $\angle$ ECF = 30° . (constr.)

Hence $\angle$ OCD = $60^\circ + 30^\circ = 90^\circ$.

Q.E.F.

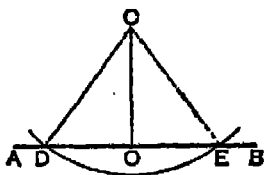


PROBLEM 3B.

To construct a perpendicular to a given straight line from a given point outside the line.

Given a straight line AB and a point C outside AB.

Required to draw from C a perpendicular to AB.



Construction. With C as centre and any sufficient radius, draw an arc cutting AB in D and E.

Bisect DE in O and join OC.

Then shall OC be *perp*^r to AB.

Proof.

Join CD, CE.

In the Δ s OCD, OCE,

$$\begin{cases} CD = CE, & (\text{constr.}) \\ OD = OE, & (\text{constr.}) \\ \text{and OC is common,} & \end{cases}$$

$\therefore$ the Δ s are congruent, (3 sides)

so that $\angle COD = \angle COE$,

$\therefore$ these $\angle$ s are right $\angle$ s, (equal adj. $\angle$ s)
and OC is *perp*^r to AB.

Q.E.F.

PRACTICAL METHOD FOR PROBLEM 3B.

In using the above construction practically it is usual to bisect DE by drawing two small arcs with centres D and E and with radius DC, to cut at P on the side of AB away from C, and then to join CP. [CP is proved to be perpendicular to AB by the same proof as in Problem 2.]

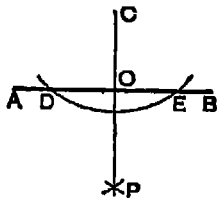
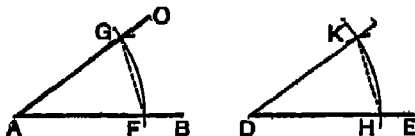


FIG. 111.

PROBLEM 4.

To construct an angle equal to a given angle at a given point in a given straight line.

Given an angle BAC and a point D in the str. line DE .
Required to construct an angle at D equal to $\angle BAC$.



Construction. With centre A and any radius draw an arc, cutting AB in F and AC in G ;

With centre D and the same radius draw an arc HK , cutting DE in H ;

With centre H and radius equal to GF draw an arc cutting arc HK in K .

Join DK .

Then *shall* $\angle D = \angle A$.

Outline of Proof. Join FG , HK .

Then we can prove $\triangle HDK$, $\triangle FAG$ congruent, (*3 sides*)

$\therefore \angle D = \angle A$.

Q.E.F.

EXAMPLES 30 (continued).

PROBLEM 2.

5. Draw a line 5·7 cm. long and bisect it with ruler and compass. Check the accuracy of your bisection by measurement.

6. Draw a different figure for Problem 2 in which $AC = BC$ and $AD = BD$, but AC is not equal to AD . Prove that $AO = OB$ in this new figure.

7. Is there any advantage in making AC , BC , AD , BD all equal (as in the Proposition) instead of equal in pairs (as in Ex. 6)?

8. Euclid bisected a straight line AB by describing an equilateral $\triangle ABC$ on it, and then bisecting the vertical angle ACB . Prove that this method is a correct one.

PRACTICAL DRAWING EXERCISES.**EXAMPLES 31.****PROBLEMS 3A, 3B, 4.**

1. Draw a straight line AB 5·7 cm. long and take a point P in it 1·5 cm. from A. Erect a perpendicular to AB at P by the second method of Problem 3A.

2. Draw a straight line AB 3" long and take a point P in it 1·2" from A. Using the first method of Problem 3A erect a perpendicular to AB at P.

3. Draw any $\triangle ABC$ having all its angles acute. Draw a perpendicular from A to BC.

4. Draw a triangle whose sides are 2", 2·4", 2·8". Draw the three lines which bisect these sides at right angles. (In a well-drawn figure these lines go through the same point.)

5. Draw a triangle whose sides are 4 cm., 5 cm., 6 cm. Draw the perpendicular from the largest angle to the opposite side and measure it.

6. Draw an angle of 77° with your protractor; copy it with the aid of ruler and compasses only; test the new angle by means of your protractor.

7. Draw with your protractor an angle of 36° , and by using your ruler and compasses make use of the angle of 36° to draw angles of (i) 72° ; (ii) 18° ; (iii) 54° . Check with your protractor.

CONSTRUCTIONS BY MEANS OF RULER AND COMPASSES.

[These exercises should be done without any use of protractor and set-square, in order to obtain accuracy in the use of compasses.]

EXAMPLES 32.

1. Draw a $\triangle ABC$ in which $AB = 3$ cm., $BC = 4$ cm., $CA = 5$ cm. Draw (in a separate figure) a horizontal line XY; make an $\angle YXZ$ (above XY) equal to $\angle A$ of $\triangle$, and an $\angle YXW$ (below XY) equal to $\angle C$. Measure $\angle ZXW$ with a protractor.

2. Draw a line AB 2·5" long. Divide it into four equal parts. Measure the four parts correct to '01".

3. Draw a $\triangle ABC$ in which $AB = 2''$, $BC = 3''$, $AC = 3·9''$. Draw a line BX from B perpendicular to AC and measure BX.

4. Draw a $\triangle ABC$ in which $BC = 2·6''$, $\angle B = 30^\circ$ and $\angle C = 90^\circ$. Measure AB and AC.

5. Draw two parallel lines AX and BY, AB being perpendicular to the lines and 2·8 cm. long. Mark off BY on the second line 2 cm. long and find a point X on AX such that $BX = 4·5$ cm. Join XY and measure it.

6. Copy the figure adjoining and see whether the base of the triangle is in a straight line with the piece to the right. (Why should it be?)

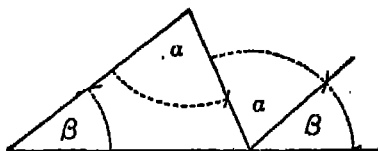


FIG. 112.

7. Draw a Δ whose sides are 8 cm., 5 cm., 7 cm. long. Bisect the three angles. Do the three bisectors go through a point?

8. Draw a ΔABC in which $AB = 2''$, $\angle A = 90^\circ$, $BC = 2.8''$. Make outside the triangle an angle BAX at A equal to $\angle B$, and an angle ABX equal to 90° . Measure AX and BX .

9. Draw a triangle ABC whose sides are 7 cm., 4 cm., 9 cm. Draw the three perpendiculars from the vertices to the opposite sides. Measure them. Do they all go through one point?

CONSTRUCTIONS (WITHOUT THE AID OF PROTRACTOR OR SET-SQUARE) OF ANGLES OF 60° , 30° , 45° .

1. To make an angle of 60° with a given line AB . (Note that 60° is the angle of an equilateral Δ .) An arc CD is drawn with A as centre, and a small arc cutting arc CD at C is drawn with the same radius and with D as centre (see Fig. 113).

If AC is joined, $\angle A$ will be 60° . (Why?)

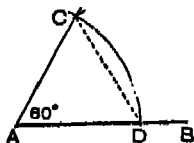


FIG. 113.

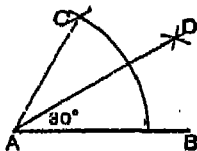


FIG. 114.

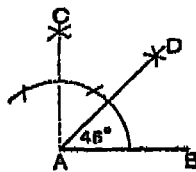


FIG. 115.

2. To make an angle of 30° with a given line AB . Make an angle 60° as above and bisect it by the method of Problem 1 (Fig. 114).

3. To make an angle of 45° with a given line AB . Make an angle of 90° and bisect it. (The figure shows the method which depends on Problem 3B.)

EXAMPLES 33.

By means of ruler and compasses construct angles of:

- (i) 30° , (ii) 45° , (iii) $22\frac{1}{2}^\circ$, (iv) 135° ,
(v) 16° , (vi) 75° , (vii) $112\frac{1}{2}^\circ$.

Measure your angles with a protractor after construction.

INEQUALITIES.

We know that:

A straight line is the shortest distance between two points.
(See page 2.)

It follows at once from this fact that:

I. Any two sides of a triangle are together greater than the third side.

Thus in a $\triangle ABC$, $BA + AC > BC$, since BC is a straight line, and BAC a bent line, connecting B and C .

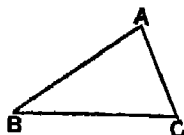


FIG. 116.

EXAMPLES 34(a).

1. Prove that any side of a triangle is greater than the difference between the other two sides.

2. Find whether it is possible to make $\triangle s$ whose sides are:

- | | |
|--------------------------|------------------------|
| (i) 3 cm., 4 cm., 5 cm.; | (ii) 1", 2", 3"; |
| (iii) 1.8", 1.5", 4"; | (iv) 3.8", 3.1", 2.1". |

3. If two sides of a triangle are 3" and 4" and the third side is an exact number of inches, state all possible lengths it can have.

4. Prove that the perimeter of a quadrilateral $>$ twice either of the diagonals.

5. In a quadrilateral $ABCD$, $AB + BC + CD > AD$.

6. Prove that the sum of the four sides of a quadrilateral is less than twice the sum of the diagonals.

7. O is a point inside $\triangle ABC$. Prove that

$$BO + OC < BA + AC.$$

[Produce BO to meet AC in D . Prove that:

$$BO + OC < BD + DC \text{ and } BD + DC < BA + AC.]$$

8. ABO and PQR are two $\triangle s$ of which the vertices of the latter lie on the sides of the former; prove that the perimeter of $PQR <$ that of ABC .

9. $ABCD$ is a quadrilateral and O is a point inside it. Prove that $OA + OB + OC + OD$ is generally $> AC + BD$. What is the one position of O for which

$$OA + OB + OC + OD = AC + BD?$$

10. In the figure to Ex. 7, prove that:

$$2(OA + OB + OC) > BC + CA + AB.$$

We have also learnt the truth of two other inequalities:

II. If one side of a triangle be produced, the exterior angle is greater than either of the interior opposite angles.

For if the side BC of $\triangle ABC$ is produced to D,
 $\angle ACD > \angle A$ and $\angle ACD > \angle B$, since
 $\angle ACD = \angle A + \angle B$ by Theorem 7.

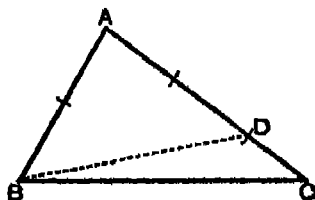
III. Any two angles of a triangle are together less than two right angles.

THEOREM 15.

If two sides of a triangle are unequal, the greater side has the greater angle opposite to it.

Given a $\triangle ABC$ in which $AC > AB$;

Required to prove that $\angle B > \angle C$.



Construction. From AC cut off AD equal to AB;
 Join BD.

Proof. In $\triangle ABD$, $\because AB = AD$, (constr.)
 $\therefore \angle ADB = \angle ABD$; (opp. = sides)
 but $\angle ABC > \angle ABD$, (whole > part)
 $\therefore \angle ABC > \angle ADB$.

Also ext^r $\angle ADB$ of $\triangle BCD >$ int^r opp. $\angle C$,
 $\therefore \angle ABC$ is much greater than $\angle C$.

Q.E.D.

THEOREM 16.

If two angles of a triangle are unequal, the greater angle has the greater side opposite to it.

Given a $\triangle ABC$, in which $\angle B > \angle C$.

Required to prove that $AC > AB$.

Proof.

AC cannot $= AB$,
for then $\angle B$ would $= \angle C$,
and this is untrue. (*given*)

Again AC cannot be less
than AB ,

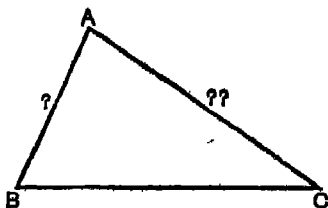
for then $\angle B$ would be
less than $\angle C$,

and this also is untrue.

$\therefore AC$ is neither equal to, nor less than, AB ,
that is, $AC > AB$.

(*given*)

Q.E.D.

**EXERCISES ON INEQUALITIES.****EXAMPLES 34(b).**

1. ABC is a $\triangle$ in which $a = 1.6''$, $b = 1.8''$, $c = 3''$. Which is the largest angle and which the smallest? Why? Check by a drawing to scale.

2. Prove that the hypotenuse is the greatest side of a right-angled triangle.

3. If, in the figure to Theorem 17, R is a point between Q and B , prove that $OR > OQ$.

4. If a $\triangle ABC$, $AB > AC$. The bisectors of the angles B and C meet in I ; prove that $BI > CI$.

5. If ABC be an equilateral triangle and D any point in AB , prove that $CD < AC$ and $CD < BC$.

6. $ABCD$ is a quadrilateral, AD is the shortest side and BC the longest. Join BD and prove that $\angle ABC < \angle ADO$.

7. ABC is a triangle and $\angle A$ is bisected by the line meeting BC in D . Prove that $AB > BD$ and $AC > CD$.

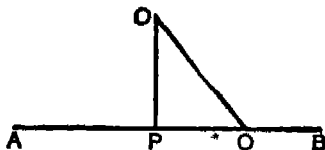
8. D is any point in the base BC of a $\triangle ABC$; prove that $AD < \frac{1}{2}$ the perimeter of the triangle.

THEOREM 17.

Of all straight lines which can be drawn to a given straight line from a given point outside it, the perpendicular is the shortest.

Given a straight line AB and a point O outside AB ;
also OP the perp^r to AB , and OQ any other line from O to AB .

Required to prove that OP is shorter than OQ .



Proof. In $\triangle OPQ$, $\angle OPQ$ is a right $\angle$,
 $\therefore \angle OQP < \text{a right } \angle$, ($\angle s \text{ of a } \triangle = 2rt. \angle s$)
 $\therefore \angle OPQ > \angle OQP$,
 $\therefore OQ > OP$; (*greater side opp. greater $\angle$*)
 that is, OP is shorter than OQ .

Similarly it may be shown that OP is shorter than any other line from O to AB .

Q.E.D.

EXAMPLES 34 (continued).

9. O is a point inside the $\triangle ABC$; prove that $\angle BOC > \angle BAC$.
10. In any triangle prove that the perimeter $>$ the sum of the perpendiculars drawn from the vertices to the opposite sides.
11. Prove Theorem 16 directly by making an $\angle CED = \angle C$ so that D is on AC .
12. In $\triangle ABC$, $AB = AC$; the base BC is produced to D and X is any point in AB . If DX cut AC in Y , prove that $AX < AY$.
13. D bisects side BC of $\triangle ABC$; prove that $AD < \frac{1}{2}(AB + AC)$.
 [Produce AD to E , making $DE = AD$; join EC and prove two $\triangle s \equiv$]
14. Use No. 13 to prove that the sum of the medians of a $\triangle <$ the perimeter.

PARALLELOGRAMS.

Definition.—A Parallelogram is a plane four-sided figure whose opposite sides are parallel (*see Figure below*).

THEOREM 18.

The opposite angles of a parallelogram are equal.

Given a par^m ABCD.

Required to prove that $\angle A = \angle C$ and $\angle B = \angle D$.



Proof. $\because$ AB and DC are par^l and AD meets them,
 $\therefore \angle A + \angle D = 2 \text{ right } \angle\text{s};$ (*int' $\angle\text{s}$*)
 similarly $\angle C + \angle D = 2 \text{ right } \angle\text{s};$
 $\therefore \angle A + \angle D = \angle C + \angle D,$
 $\therefore \angle A = \angle C.$
 Similarly $\angle B = \angle D.$

Q.E.D.

***Conversely.**—If the opposite angles of a quadrilateral are equal, the figure is a parallelogram.

Given a quad^l ABCD, in which $\angle A = \angle C$ and $\angle B = \angle D$.

Required to prove that ABCD is a par^m .

Proof. $\angle\text{s A, B, C, D together} = 4 \text{ rt. } \angle\text{s};$ ($\angle\text{s of a quad}^l$)
 but $\angle A = \angle C$ and $\angle B = \angle D,$ (*given*)
 $\therefore \text{twice } \angle C + \text{twice } \angle D = 4 \text{ right } \angle\text{s},$
 $\therefore \angle C + \angle D = 2 \text{ right } \angle\text{s},$
 $\therefore$ AD and BC are $\text{par}^l.$ (*int' $\angle\text{s supp}^v$*)

Similarly it may be shown that

AB and CD are $\text{par}^l;$

$\therefore$ ABCD is a $\text{par}^m.$

(*definition*)

Q.E.D.

* For the meaning of the Converse of a Theorem, see page 107.

THEOREM 19.

The opposite sides of a parallelogram are equal, and each diagonal bisects the area of the parallelogram.

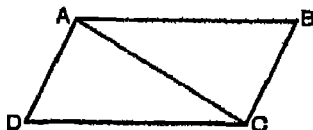
Given a par^m ABCD of which the diagonal AC is drawn.
 Required to prove that (i) $AD = BC$; (ii) $CD = AB$,
 and (iii) $\triangle ADC = \triangle ABC$ in area.

Proof. In $\triangle s$ DAC, BCA,
 $\left\{ \begin{array}{l} \angle DAC = \angle BCA, \quad (*alt.) \\ \angle DCA = \angle BAC, \quad (*alt.) \end{array} \right.$
 and AC is common,

$\therefore \triangle s$ are congruent,

(2 $\angle s$ and a side)

so that $AD = BC$, $CD = AB$,
 and $\triangle DAC = \triangle BCA$ in area.



Q.E.D.

Conversely (to first part of above).—If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.

Given a quad^l ABCD in which $AB = DC$ and $AD = BC$.
 Required to prove that ABCD is a par^m .

Construction. Join AC.

Proof.

In $\triangle s$ DAC, BCA,

$\left\{ \begin{array}{l} AD = BC, \quad (given) \\ CD = AB, \quad (given) \\ \text{and AC is common,} \end{array} \right.$

$\therefore \triangle s$ are congruent, (3 sides)

so that $\angle DAC = \angle BCA$,

$\therefore AD$ and BC are par^l ; (alt. $\angle s =$)

similarly it may be shown that AB and CD are par^l ,

$\therefore$ ABCD is a par^m . (definition)

Q.E.D.

Note.—The first part of Theorem 18 can be, and often is, proved by the method of Theorem 19. Write out this way of proving it.

* The word "*alternate*" is used here to include the fact that lines are $\parallel$; otherwise alternate angles would not be equal, and so throughout this book. Similarly with the word "*corresponding*."

THEOREM 20.

The diagonals of a parallelogram bisect one another.

Given a par^m ABCD, its diagonals meeting in E.

Required to prove that $AE = EC$, and $DE = EB$.



Proof.

In Δs DAE, BCE,
 $\begin{cases} \angle DAE = \angle BCE, & (\text{alternate}) \\ \angle AED = \angle CEB, & (\text{vert. opp.}) \\ \text{and } AD = BC, & (\text{opp. sides of } \text{par}^m) \end{cases}$
 $\therefore \Delta s$ are congruent; $(2 \angle s \text{ and a side})$
 so that $AE = EC$,
 and $DE = EB$.

Q.E.D.

Conversely.—If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.

Given a quad^l ABCD, its diagonals meeting in E, so that $AE = EC$ and $DE = EB$.

Required to prove that ABCD is a par^m .

Proof.

In Δs DAE, BCE,
 $\begin{cases} AE = EC, & (\text{given}) \\ DE = EB, & (\text{given}) \\ \text{and } \angle AED = \angle CEB, & (\text{vert. opp.}) \end{cases}$
 $\therefore \Delta s$ are congruent, $(2 \text{ sides and incl. } \angle)$
 so that $\angle DAE = \angle BCE$,
 $\therefore AD$ and BC are par^l ; $(\text{alt. } \angle s =)$
 similarly it may be shown that AB and DC are par^l ,
 $\therefore ABCD$ is a par^m . (definition)

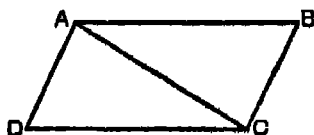
Q.E.D.

THEOREM 21.

If one pair of opposite sides of a quadrilateral are equal and parallel, the figure is a parallelogram.

Given a quad^l ABCD, in which $AB = CD$ and AB is par^l to CD .

Required to prove that ABCD is a *par^m*.



Construction. Join AC.

Proof.

In Δ s ABC, CDA,

$AB = CD$,

(given)

AC is common,

and $\angle BAC = \angle DCA$,

(alternate)

$\therefore \Delta$ s are congruent, (2 sides and incl. $\angle$)

so that $\angle BCA = \angle DAC$,

$\therefore AD$ and BC are par^l.

(alt. $\angle$ s =)

But AB and CD are also par^l,

(given)

$\therefore ABCD$ is a *par^m*.

(definition)

Q.E.D.

NOTE ON THEOREMS 18-21.

We now have five ways of testing whether a quadrilateral is a parallelogram, viz :

- (i) If the opposite sides are parallel ; (by the definition)
- or (ii) If the opposite angles are equal ; (by Theorem 18)
- or (iii) If the opposite sides are equal ; (by Theorem 19)
- or (iv) If the diagonals bisect one another ; (by Theorem 20)
- or (v) If one pair of opposite sides are equal and parallel.
(by Theorem 21)

FURTHER DEFINITIONS.

A rectangle is a parallelogram which has *one* of its angles a right angle.

[It follows that all the angles are right angles. See Ex. 1 on page 98.]

A square is a rectangle which has *two adjacent* sides equal.

[It follows that all the angles of a square are right angles and all its sides are equal. See Exx. 1 and 2 on page 98.]

A rhombus is a parallelogram with *two adjacent* sides equal.

[It follows that all the sides of a rhombus are equal. See Ex. 2 on page 98.]

A trapezium is a quadrilateral which has one pair of opposite sides parallel.

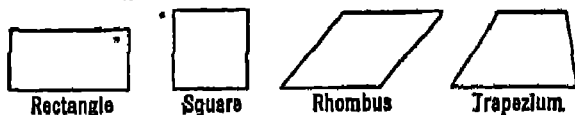


FIG. 117.

Important Facts.

1. The diagonals of a rectangle are equal. [See Ex. 12 on page 98.]

2. The diagonals of a rhombus bisect each other at right angles.

[They bisect each other because a rhombus is a parallelogram; they cut at right angles because the Δ s ABO, CBO are congruent (*three sides*). See clearly why the sides of the Δ s are equal.]

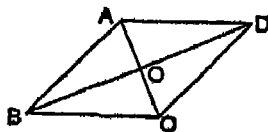
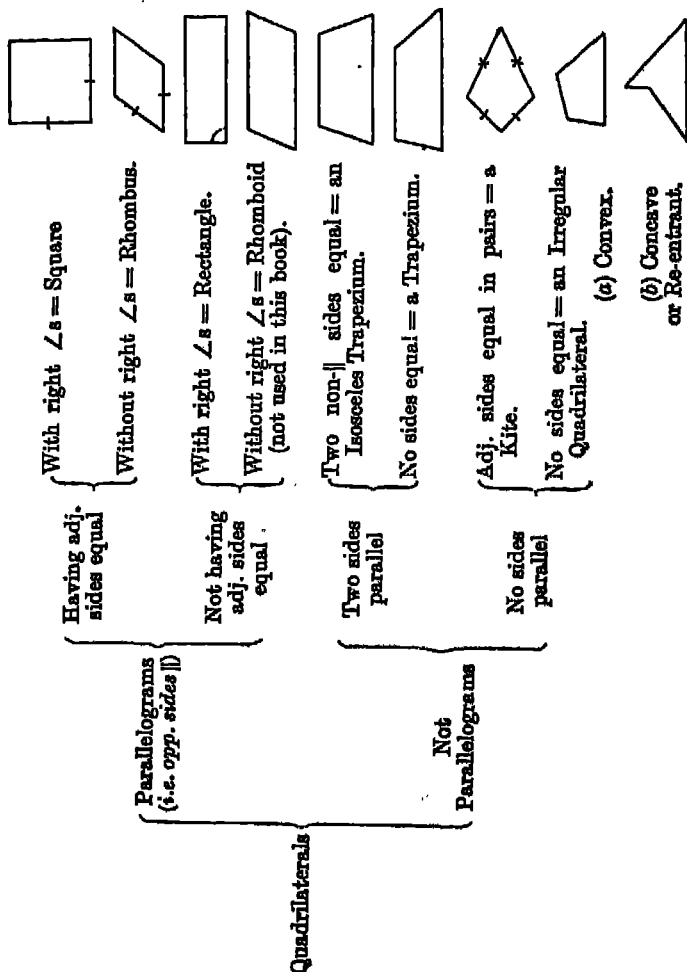


FIG. 118.

3. The diagonals of a square are equal and bisect each other at right angles. [This follows from 1 and 2 above.]

4. The only type of parallelogram in which a diagonal bisects the angles which it connects is the rhombus (including, of course, the square). [See Ex. 5 on page 98.]

Types of Quadrilaterals.



EXERCISES ON PARALLELOGRAMS.

EXAMPLES 35.

1. Prove that, if one angle of a parallelogram is a right angle, all its angles must be right angles.

[Hence all the angles of a rectangle are right angles; compare definition on page 96.]

2. Prove that if two adjacent sides of a parallelogram are equal, all its sides are equal.

[Hence all the sides of a rhombus (or a square) are equal; compare definitions on page 96.]

3. If one angle of a parallelogram is 35° , find the sizes of the other three angles.

4. ABCD is a parallelogram; E bisects AB and F bisects DC. Prove that AECF is also a parallelogram.

5. If the diagonal AC of a parallelogram bisects the $\angle A$, prove that it also bisects $\angle C$ and that the figure has its four sides equal.

6. ABCD is a parallelogram and X is a point on AB and Y a point on DC such that $AX = YC$. Prove that AY and XC, when joined, are equal and parallel.

7. Prove that the bisectors of two opposite angles of a parallelogram are parallel unless they happen to coincide.

8. Show that the bisectors of two adjacent angles of a parallelogram form a right angle.

9. Show that two congruent triangles can be placed with one side common so as to form a parallelogram.

10. ABCD is a parallelogram whose diagonals meet in O; a line is drawn through O to meet AB in X and CD in Y. Show by means of congruent triangles that $OX = OY$.

11. Prove in the figure to Ex. 10 that XY bisects the parallelogram.

12. Show that the diagonals of a rectangle are equal.

13. Prove that the diagonals of a rectangle divide it into four isosceles triangles.

14. Prove that the diagonals of a rhombus (or of a square) bisect the angles through which they pass.

15. ABCD is a parallelogram. Points P, Q, R, S are taken in AB, BC, CD, DA, such that $AP = CR$; $BQ = DS$. Prove that PQRS is a parallelogram.

16. Show by drawing figures what different kinds of quadrilaterals may be formed if one pair of straight railway lines crosses another pair.

Consider the two cases (i) when the gauge (i.e. the perpendicular distance between the metals) is different, and (ii) when the gauge is the same for the two pairs.

17. ABCD is a parallelogram and through O, the intersection of its diagonals, a line is drawn meeting AB in P and CD in Q. Prove that APCQ is a parallelogram.

18. State how you would test (by means of its diagonals only) whether a quadrilateral is:

(i) a parallelogram;

(iii) a square;

(ii) a rectangle;

(iv) a rhombus.

19. ABCD is a parallelogram and points X, Y, Z, W are taken on the sides AB, BC, CD, DA respectively so that $AX = CZ$ and $AW = CY$. Prove that XZ and WY bisect each other. [For this to be true, what sort of figure must XYZW be? Set out to prove this first of all.]

20. In an isosceles trapezium (i.e. one in which the non-parallel sides are equal) prove that:

(i) the angles are equal in pairs; (ii) the diagonals are equal.

21. ABCD is a par^m. Through C is drawn a par^l to the bisector of $\angle B$ meeting AB, AD produced in X, Y. Prove that $\triangle AXY$ is isosceles and that perimeter of par^m is $AX + AY$.

22. A square BCDE is described on the side BC of $\triangle ABC$, and through B, C are drawn lines BF, CF $\parallel$ to AE, AD, meeting in F. Prove that $FA \perp BC$.

TO DRAW A LINE THROUGH A GIVEN POINT (A) PARALLEL TO A GIVEN STRAIGHT LINE (BC).

A.—WITH RULES AND COMPASSES.

I. Draw a perpendicular AD from A to BC (Prob. 3B), and erect at A a perpendicular AE to AD (Prob. 3A). (See Fig. 119.)
[Why is AE parallel to BC?]

II. Draw a straight line AD from A to meet BC anywhere in D; produce DA to E, and make an angle EAF equal to, and corresponding to, ADC (Prob. 4). (See Fig. 120.)
[Why is AF parallel to BC?]

A variation of this method is to make alternate angles equal.

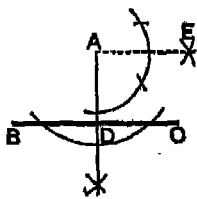


FIG. 119.

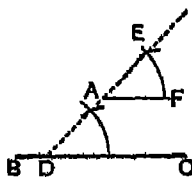


FIG. 120.

In Fig. 120 the arc with centre D should have the small arc intersecting it between D and C, not between D and A.

III. Remember that a quad^l is a par^m if its opposite sides are equal, and try to make such a quad^l ABCD, given A and BC, so that D only has to be found. Hence:

With centre A and radius BC draw an arc, and with centre C and radius BA draw an arc; let the arcs cut in D.

[Why is ABCD a par^m?]

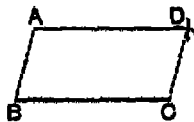


FIG. 121.

B.—WITH SET-SQUARES.

I. Method A I. can still be used, set-squares being used in order to draw the right angles.

II. The ordinary method of using a set-square to draw parallels is found on p. 17.

DRAWING EXERCISES ON PARALLELOGRAMS, ETC.

EXAMPLES 36.

1. Draw a square on a line 2" long, and measure its diagonals.
2. On a line 4 cm. long draw a rectangle 2.5 cm. broad, and measure its diagonals.
3. Describe a rhombus having each side 4 cm. long, and containing an angle of 45° . Measure its diagonals.
4. Draw a rhombus having each side equal to one of its diagonals. How many degrees are there in each angle?
5. On a line 1.8" long as diagonal describe a square. Measure one of the sides.
6. On a line 5 cm. long describe a rhombus having one diagonal 6.5 cm. long. Measure the other diagonal.
7. Draw a parallelogram with sides 4 cm. and 5 cm. long, containing an angle of 52° . Measure the longer diagonal.
8. On a base 2.5 cm. long describe an isosceles triangle ABC, having each of the base angles 72° . Complete the parallelograms of which the sides of triangle ABC give two sides and one diagonal. In how many ways can this be done?
9. Draw a quadrilateral ABCD in which $AB = CD = 7$ cm., $AD = BC = 5$ cm., and diagonal $AC = 10$ cm. Measure its angles.
10. Draw a parallelogram whose diagonals are 2" and 2.8", the obtuse angle between them being 120° . Measure the sides.
11. Draw a rectangle whose diagonals are 7 cm. long, the acute angle between them being 70° . Measure the sides.
12. Draw a rhombus whose diagonals are 4 cm. and 5.5 cm. long. Measure the angles.
13. Construct a trapezium whose parallel sides are 5.4 cm. and 9.5 cm., and whose angles at the end of the latter side are 80° and 54° . (See Fig. 122.)
14. Construct a trapezium whose parallel sides are 4", 2.4", and whose non-parallel sides are 2" and 2.8". (Make use of dotted line in figure.)
15. Construct a trapezium whose parallel sides are 2.2" and 3.6", and whose diagonals are 3" and 4".
16. Without using the protractor, construct a quadrilateral ABCD, in which $AB = 4$ in., $CD = 3$ in., $DC \parallel AB$, and each of the angles DAB, CBA is 60° . Justify your construction.



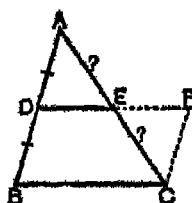
FIG. 122.

THEOREM 22.

The straight line drawn through the middle point of one side of a triangle parallel to another side bisects the third side.

Given a $\triangle ABC$ in which D is the middle point of AB , and DE is drawn par^l to BC to meet AC in E .

Required to prove that $AE = EC$.



Construction. Draw CF from C par^l to BA , to meet DE produced in F .

Proof.

EF is a par^m , (*opp. sides par^l*)

$\therefore BD = CF$; (*opp. sides of par^m*)

but $BD = DA$, (*given*)

$\therefore DA = CF$.

Then in $\triangle ADE$, CFE ,

$\begin{cases} \angle A = \angle FCE, & (\text{alternate}) \\ \angle AED = \angle CEF, & (\text{vert. opp.}) \\ \text{and } DA = CF. & (\text{proved}) \end{cases}$

$\therefore \triangle s$ are congruent, (*two $\angle s$ and a side*)
so that $AE = EC$.

Q.E.D.

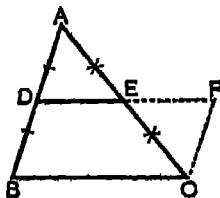
This theorem is a special case of Theorem 24, but it is a good thing to prove each independently of the other.

THEOREM 23.

The straight line joining the middle points of two sides of a triangle is parallel to the third side, and equal to one half of it.

Given a $\triangle ABC$ in which D and E , the middle points of AB and AC , are joined.

Required to prove that (i) DE is *par^l* to BC ,
and (ii) $DE = \text{half of } BC$.



Construction. Draw CF from C *par^l* to BA , to meet DE produced in F .

Proof. (i) In $\triangle s$ ADE , CFE ,

$$\begin{cases} \angle A = \angle FCE, & (\text{alternate}) \\ \angle AED = \angle CEF, & (\text{vert. opp.}) \\ \text{and } AE = EC, & (\text{given}) \end{cases}$$
 $\therefore \triangle s$ are congruent, (*two $\angle s$ and a side*)
 so that $AD = CF$;
 but $BD = DA$, (*given*)
 $\therefore BD = CF$.
 Now BD and CF are also *par^l*, (*constr.*)
 $\therefore BDFC$ is a *par^m*,
 that is, DE is *par^l* to BC .

(ii) $DE = EF$, (*congruent $\triangle s$*)
 and $DF = BC$, (*opp. sides of par^m*)
 $\therefore DE = \frac{1}{2} DF = \frac{1}{2} BC$.

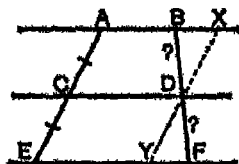
Q.E.D.

THEOREM 24.

If three or more parallel straight lines make equal intercepts on any transversal, they make equal intercepts on any other transversal.

Given three par^l lines AB, CD, EF, which are cut by the transversals ACE, BDF, so that $AC = CE$.

Required to prove that $BD = DF$.



Construction. Draw XDY through D par^l to AE, to meet AB produced in X and EF in Y.

Proof. Then the figures CX and CY are par^{ms},

$$\therefore AC = XD \text{ and } CE = DY; \quad (\text{opp. sides})$$

$$\text{but } AC = CE, \quad (\text{given})$$

$$\therefore XD = DY.$$

Now in Δs BXD, FYD,

$$\begin{cases} \angle DBX = \angle DFY, & (\text{alternate}) \\ \angle BDX = \angle FDY, & (\text{vert. opp.}) \\ \text{and } XD = DY, & (\text{proved}) \end{cases}$$

$$\therefore \Delta s \text{ are congruent, } (\text{two } \angle s \text{ and a side})$$

$$\text{so that } BD = DF.$$

Q.E.D.

MISCELLANEOUS EXERCISES ON PARALLELS AND INTERCEPTS.

EXAMPLES 37.

1. The straight lines which join the middle points of the sides of a triangle divide it into four congruent triangles.

2. AB, AC, AD are three straight lines drawn from A, and their middle points are X, Y, Z. Prove that $\angle XYZ = \angle BCD$.

3. The line joining the middle points of two sides of a triangle will bisect the line drawn from any point on the third side to the opposite vertex.

4. ABC is a triangle and D a point on BC. If the middle points of AB, AC, BD, DC are P, Q, R, S, prove that PQSR is a parallelogram.

5. ABC is a Δ having the side AB trisected at D and E. Prove that the straight lines drawn through D and E parallel to BC will trisect AC and that the shorter of them (terminated by AB and AC) is $\frac{1}{3}$ BC in length.

6. If the middle points of adjacent sides of a quadrilateral be joined, the resulting figure is a parallelogram.

7. Prove that the two straight lines which join the middle points of opposite sides of any quadrilateral bisect each other. (See Ex. 6.)

8. ABCD is a field with four straight sides, and its diagonals AC and BD are 65 yds. and 43 yds. long. If P, Q, R, S are stakes placed at the centre of the four sides of the field, find the total length round PQRS.

9. If the middle points of the non-parallel sides of a trapezium are joined, the joining line is parallel to each of the parallel sides of the figure. [Converse to Theorem 24.]

10. ABC is a line bisected at B, and AX, BY, CZ are drawn perpendicular to a line XYZ. Prove that, if A and C are on the same side of XYZ,

(i) $BY - AX = CZ - BY$, if AX is the shortest of the perpendiculars;

(ii) $BY = \frac{1}{2}(AX + CZ)$.

11. In the figure to Theorem 24, prove that $CD = \frac{1}{2}(AB + EF)$.

12. ABCD is a parallelogram and through B a straight line XY is drawn outside the figure. The perpendiculars from A and from the mid-point of BD on XY are $1\frac{1}{2}$ in. and $2\frac{1}{2}$ in. respectively. Calculate the perpendiculars from D and C on XY.

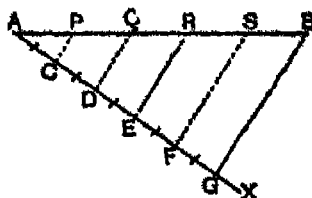
PROBLEM 5.

To divide a straight line into any number of equal parts.

Given a straight line AB .

Required to divide it into any number of equal parts (say five).

Method I.



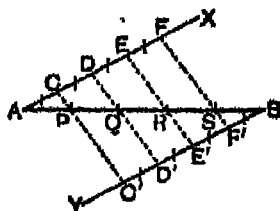
Construction. From A draw AX making any angle with AB ; mark off along AX five equal lengths, viz., AC , CD , DE , EF , FG ; join GB .

Draw lines $\parallel$ to GB through C , D , E , F , to meet AB in P , Q , R , S . Then shall $AP = PQ = QR = RS = SB$.

Proof. This follows at once from Theorem 24.

Q.E.F.

Method II.



Construction. From A and B draw two parallel lines AX , BY , on opposite sides of AB ;

mark off equal lengths along each as shown.

Join CC' , DD' , EE' , FF' , meeting AB in P , Q , R , S .

Then shall $AP = PQ = \text{etc.}$

Proof. This follows from Theorems 21 and 24, CC' , DD' etc. being easily shown to be parallel.

Q.E.F.

EXAMPLES 38—PRACTICAL.

1. Draw a line 3 inches long and divide it into seven equal parts.
2. Draw $\frac{3}{4}$ of a line which is 5.4 inches long. Measure its length and check your result by arithmetic.

THEOREMS AND THEIR CONVERSES.

Theorems like those numbered 11 and 12 (see pages 74 and 75) are said to be converse to each other, and Theorem 12 can be called the converse of Theorem 11. In such a pair of Theorems what is *proved in one is assumed as true in the other*, and vice versa. Thus in Theorem 11, the angles B and C are proved equal, but in Theorem 12 these angles are assumed equal. So also the sides AB and AC are assumed equal in Theorem 11 and proved equal in Theorem 12.

Theorems 15 and 16 are also a pair of Converse Theorems. Find other pairs of Converse Theorems in this book.

It must not be assumed that the converse to every proposition or rider is true.

Thus it is true that "*if two triangles have the sides of one equal to the sides of the other, each to each, the corresponding angles of the triangles are equal.*"

The converse statement is that "*if two triangles have the angles of one equal to the angles of the other, each to each, the corresponding sides of the triangles are equal.*"

Is this true?

The converse of a proposition or rider must never be assumed to be true until it has been proved to be correct.

State the converse statements to the following. Can you see at a glance that any of the converses are untrue?

- (a) *If a triangle is equilateral, it is also isosceles.*
- (b) *If a quadrilateral is a parallelogram, a diagonal bisects its area.*
- (c) *If a line is drawn through the point of intersection of the diagonals of a parallelogram, it will bisect the area of the figure.*
- (d) *If two Δ s ABC, DEF are congruent (taking corresponding angles in the order given) and X, Y are the middle points of BC, EF, then $AX = DY$.*

On the other hand the converses of many Theorems are true. See for instances Theorems 18, 19, 20 and their converses.

SECTION II.

AREA.

The area of a figure is expressed in terms of *a unit of area*. The units most often used in Geometry are the *square inch* (1 sq. in. or 1 in².) and the *square centimetre* (1 sq. cm. or 1 cm²).

An approximate method of finding the area of any figure is by ruling lines $\frac{1}{10}$ inch apart across it in two perpendicular directions. This divides the figure up into whole squares and parts of squares. The area of each square will be $\frac{1}{100}$ sq. inch. The number of complete squares inside the figure is then counted up, and then also each square is counted which has more than half its area inside the figure which is being dealt with. All squares are ignored which have less than half their area inside the figure.

Instead of dividing up the area into small squares by ruling lines across it, a very laborious method, the area may be traced out on squared paper, or a piece of transparent squared paper may be put over the area. In each case the squares to be counted and the given figure are seen together at a glance.

The above is the usual method of finding the area of any figure which is irregular in shape. In the case of a regular figure some formula can generally be found, such as the formula given below for the area of a rectangle.

AREAS OF SIMPLE FIGURES.

It is a well-known fact in Arithmetic that:

The area of a rectangle a feet long and b feet wide is ab sq. ft.

This is illustrated by the figure below where $a = 5$ and $b = 3$ and each square represents 1 sq. ft.

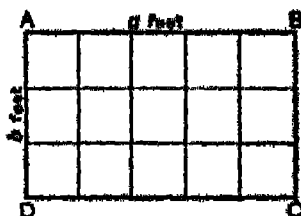


FIG. 134.

$$AB = 5 \text{ ft.}$$

$$AD = 3 \text{ ft.}$$

$$\text{Area of rectangle} = 3 \times 5 = 15 \text{ sq. ft.}$$

In general, taking the case in which a and b are whole numbers, the rectangle can be divided into b rows of squares, each row containing a squares.

$$\therefore \text{the area of the rectangle} = a \times b \text{ sq. ft.} \\ = ab \text{ sq. ft.}$$

The same fact is true when a and b are fractions.

To take a simple case:

$$\text{If } a = 2\frac{1}{2} \text{ ft., } b = 1\frac{1}{2} \text{ ft.}$$

Divide up a into tenths of a foot, 25 in all; also divide up b into tenths of a foot, 12 in all.

Then, dividing up the area, as before into squares (each side $\frac{1}{10}$ foot), we see that the number of squares in the area is 25×12 .

But the size of each square is $\frac{1}{100}$ sq. ft., for a square foot could be divided up into 10×10 squares each measuring $\frac{1}{10}$ ft. $\times$ $\frac{1}{10}$ ft. (see figure 135).

$$\therefore \text{the area of the rectangle} \\ = 25 \times 12 \text{ hundredths of a sq. ft.} \\ = 25 \times 12 \text{ sq. ft.} \\ = 25 \times 12 \text{ sq. ft.} = 3 \text{ sq. ft.}$$

We thus learn the truth of Theorem 25 (see next page).

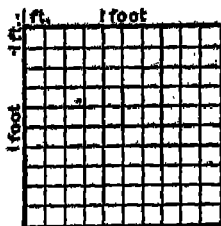


FIG. 135.

THEOREM 25.

The area of a rectangle is measured by the product of the measures of its sides.

This is generally stated more briefly as:

$$\text{Area of a rectangle} = \text{length} \times \text{breadth.}$$

It follows that:

$$\text{The area of a square} = \text{side} \times \text{side} = (\text{side})^2.$$

The area of a rectangle contained by AD and DC can be written AD, DC.

The area of a square on the side AB can be written AB².

Two figures whose areas are equal are described as equivalent.

Triangles which are equivalent need not be congruent.

So too with figures. For example, a rectangle measuring 9" $\times$ 4" and a square whose side is 6" are equivalent (why?); they are however not congruent.

The altitude (or height) of a parallelogram on a given base is the perpendicular distance between the base and the opposite side. (See Fig. 136.)

Since any side may be taken as base, a parallelogram has two altitudes. (Why not four?)



FIG. 136.

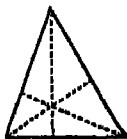


FIG. 137.



FIG. 137A.

The altitude (or height) of a triangle is the perpendicular drawn from the vertex to the base. (See Fig. 137.)

Since any side may be taken as base, a triangle has three altitudes.

When one angle of a triangle is *obtuse*, notice that two of the three altitudes are *outside* the triangle.

(One such altitude is shown in Fig. 137A.)

PRACTICAL AND NUMERICAL EXERCISES ON RECTANGLES.

EXAMPLES 40.

1. Draw on squared paper (each square $\frac{1}{16}$ " side) a circle whose radius is 2". Find its area by counting squares.
2. Draw on squared paper a rectangle whose length is 1.8" and whose breadth is 1.4". Find its area (i) by counting squares; (ii) by calculation.
3. Find in two ways the area of a rectangle which measures 2.3" by 1.9".
4. Calculate the area of a rectangle 8.1 cm. long and 4.6 cm. wide.
5. Find the breadth of a rectangle if its area = 42 sq. cm. and its length is 10.5 cm. Draw it on squared paper. Check its area by counting squares.
6. The plan of a rectangular garden measures 11.5" $\times$ 8.5" when drawn on a scale of 1" to 20 feet. Calculate its area.
7. Draw a diagram to show that 1 sq. yard = 9 sq. feet.
8. The length of a rectangle is twice its breadth and when drawn on a scale of 1 cm. to 10 metres its area is 60 sq. cm. Find the actual dimensions and area of the rectangle.

Find the areas of the shaded parts in the following diagrams:

9.



Width through-
out = 3".

FIG. 138.

10.

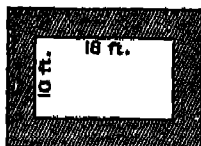


FIG. 139.

11.

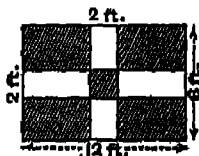


FIG. 140.

12. Prove by the method of Theorem 25 that the area of a rectangle = length $\times$ breadth in the following cases:

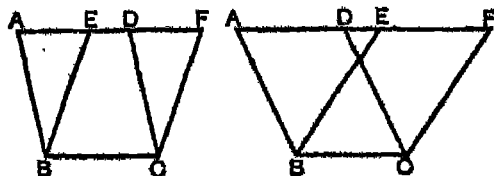
- (i) length = 6.2", breadth = 4.7";
- (ii) length = 3.4 cm., breadth = 2.8 cm.

THEOREM 26.

The areas of parallelograms on the same base and between the same parallels are equal.

Given two par^m ABCD and EBCF on the same base BC, and between the same $\parallel$ s AF and BC.

To prove that par^m ABCD = par^m EBCF in area.



Proof.

In Δ s ABE, DCF,

$$\begin{cases} \angle A = \text{corresp. } \angle FDC, & (\text{AB and DC par}) \\ \angle AEB = \text{corresp. } \angle F, & (\text{EB and FC par}) \\ \text{and AB} = \text{DC}, & (\text{opp. sides of par}^m) \end{cases}$$

$\therefore \Delta$ s $\cong$,

(two $\angle$ s and a side)

that is, Δ ABE = Δ DCF in area.

If these Δ s are taken in turn from the whole figure ABCF, the remainders are equal,

and $\therefore \text{par}^m$ ABCD = par^m EBCF in area.

Q.E.D.

Cor. 1. The area of a parallelogram is equal to the area of a rectangle on the same base and between the same parallels.

This can be deduced from Theorem 26 or proved directly, using the first figure above, but making EBCF a rectangle.

Cor. 2. The area of a parallelogram is measured by the product of the measures of its base and its altitude.

This is generally stated more briefly thus:

Area of a Par^m = base $\times$ altitude.

[N.B.—Try to prove the Δ s ABE, DCF congruent, beginning with AB = DC and EB = FC. (opp. sides of par^m)

Can you complete this (i) by the three sides congruence?

(ii) by the two sides and incl. $\angle$ congruence?

Is either method as easy as the one given in the above proof?

EXERCISES ON AREAS OF PARALLELOGRAMS.

EXAMPLES 42.

NUMERICAL AND THEORETICAL.

1. Calculate the area of each of the following parallelograms, and draw a rough sketch of each:

- (i) Base 12 cm., altitude 4.7 cm.;
- (ii) Side 1.7", perpendicular between it and opposite side 3".

2. Prove that, of all parallelograms on the same base and between the same parallels, the rectangle has the least perimeter. [Consider the rectangle and one parallelogram.]

3. ABCD is a parallelogram. X bisects AB and Y bisects CD. If XY be joined, prove that:

- (i) XY divides the figure into two parallelograms,
- and (ii) these parallelograms are equal in area.

4. ABCD and EFGH are parallelograms between the same parallels and standing on equal bases BC and FG. If EB and HC are joined, prove that:

- (i) EBCH is a parallelogram,
- and (ii) EBCH is equal to either AC or EG in area.

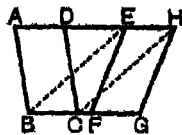


FIG. 141.

5. In Fig. 142 the lines GH and EF are parallel to the sides of the par^m AC, and they meet on the diagonal BD.

- Prove that (i) $\triangle ABD = \triangle BDC$ in area;
(ii) par^m EH = par^m GF in area.

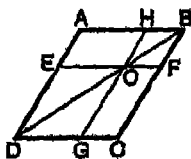


FIG. 142.

PRACTICAL.

6. Draw a parallelogram whose adjacent sides are 5 cm. and 6 cm., the angle between them being 80° . Draw and measure an altitude, and use it to calculate the area of the figure.

7. Find by drawing (and measuring the height) the area of a parallelogram whose sides are 2" and 1.3", with one diagonal 2.6".

8. On a line 5 cm. long describe a parallelogram whose area is 12 sq. cm., one of its angles being 72° . (First calculate the altitude.)

9. Draw a parallelogram ABCD in which AB = 3 cm., BC = 6 cm., and whose area is 9 sq. cm. Measure $\angle ADC$.

10. Construct a parallelogram ABCD in which AB = 4 cm., BC = 3.5 cm., and $\angle B = 50^\circ$; make a rhombus with sides 3.5 cm. equal in area to ABCD and measure the altitude of the rhombus.

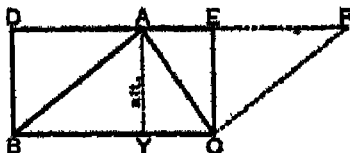
(SUGGESTION.)

THEOREM 27.

The area of a triangle is equal to one-half of the area of a rectangle on the same base and between the same parallels.

Given a $\triangle ABC$ and a rectangle $DBCE$ on the same base BC and between the same parallels BC and AE .

To prove that $\triangle ABC$ is half rectangle $DBCE$ in area.



Constr. Draw $CF \parallel BA$, completing the par^m $ABCF$.

Proof. AC is a diag. of the par^m BF ,

$\therefore \triangle ABC = \text{half of the par}^m BF \text{ in area. (diag. bisects par}^m.)$

But par^m $BF = \text{rect. } BE$, (same base BC and same $\parallel$ s)

$\therefore \text{area of } \triangle ABC = \text{half of the area of the rect. } BE.$

Q.E.D.

Cor. 1. The area of a triangle is measured by one-half of the product of the measures of its base and its altitude.

That is: Area of a $\triangle = \frac{1}{2} \text{ base} \times \text{altitude.}$

For Area of $\triangle ABO = \frac{1}{2}$ of area of rect. BE .

$$= \frac{1}{2} BC \times BD$$

$$= \frac{1}{2} BC \times AY$$

$$= \frac{1}{2} \text{ base} \times \text{altitude.}$$

Cor. 2. Triangles on the same or equal bases and of the same altitude are equal.

For each triangle has the product $\frac{1}{2} \text{ base} \times \text{altitude}$ the same.

Cor. 3. (i) If triangles of the same area have the same or equal bases, their altitudes are equal; in particular, (ii) if two triangles of the same area stand on the same side of the same base (or of equal bases in the same straight line), they are between the same parallels.

Cor. 3. (i): 1st base $\times$ 1st alt. = 2nd base $\times$ 2nd alt.;

but the bases are equal,
 $\therefore$ 1st alt. = 2nd alt.

Cor. 3. (ii): $AX = DY$, (just proved)
 but $AX \parallel DY$, (corresp. $\angle$ s equal)

$\therefore$ $AXYD$ is a par^m, (one pair of sides = and $\parallel$)
 and $\therefore AD \parallel XY$ (or BC).

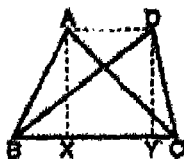


FIG. 143.

Definition. A median of a triangle is a straight line drawn from a vertex to the mid-point of the opposite side.

EXERCISES ON AREAS OF TRIANGLES.

EXAMPLES 43.

ARITHMETICAL AND THEORETICAL.

1. Find the area of a triangle given that:

(i) Base = 6.4", height = 2.5";

(ii) Base = 7.2 cm., height = 11 cm.

2. Find the altitude of a triangle given that:

(i) Area = 8.5 sq. cm., base = 1.7 cm.;

(ii) Base = 3.6", area = 2.34 sq. in.

3. $\triangle ABC$ is a $\triangle$ in which AD and BE are two altitudes. If $BC = 2.5"$, $CA = 1.6"$, $AD = 2.4"$, calculate the area of the $\triangle$ and the length of BE .

4. If a, b, c are the lengths of the sides of a triangle and x, y, z the lengths of the perpendiculars drawn to the sides from the opposite angles, prove that $ax = by = cz$.

5. Prove that the area of a $\triangle ABC$ is $\frac{1}{2} AB \cdot AC \sin \angle A = 90^\circ$.

6. Prove that a median of a triangle bisects its area.

7. Prove that a parallelogram is divided into four equal triangles by its two diagonals.

8. If two straight lines AB and CD cross in E , and $\triangle AEO = \triangle BED$ in area, prove that AD is parallel to BC .

9. $\triangle ABC$ is any $\triangle$; show how to describe several triangles on AB as base whose areas are all twice that of $\triangle ABC$. Where do their vertices lie?

10. If a quadrilateral is bisected by each of its diagonals, it is a parallelogram.

11. ABC is a Δ , and D bisects AB and E bisects AC ; join DE , BE and CD ; prove that:

(i) ΔBDE and ΔCED each $= \Delta ADE$ in area.

(ii) DE is parallel to BC .

12. Prove, by means of areas, that the perpendiculars from A and C on the diagonal BD of a parallelogram $ABCD$ are equal.

13. AX is a median of ΔABC and O is any point in AX ; prove that $\Delta ABO = \Delta ACO$.

(This can be proved either with or without the altitudes shown on the figure.)

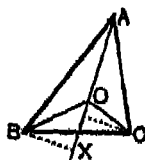


FIG. 144.

14. If ABD and ACD are equal Δ s on opposite sides of the same base AD , prove that BC , the line joining the vertices, is bisected by the base AD (or the base produced). (Drop perpendiculars from B and C to AD .)

15. $ABCD$ is a quadrilateral whose area is bisected by the diagonal AC ; prove that the diagonal AC bisects the diagonal BD .

PRACTICAL.

16. Draw a Δ in which $a = 10$ cm., $b = 8$ cm., $c = 5$ cm. Measure one of the altitudes and calculate the area of the Δ .

17. Draw a Δ whose sides are $2.5''$, $1.8''$, $3''$. Find three values for the area of the Δ (taking each side in turn as base) and find the average of the three results (correct to $.01$ sq. inches).

18. On a base of $3''$ construct an isosceles Δ whose area is 2.4 sq. ins. (Calculate the height first of all.)

19. On a given straight line AB , $1.2''$ long, construct (i) a right-angled Δ ; (ii) an isosceles Δ , each having an area of $.9$ sq. ins.

20. Construct a Δ , given that the area $= 2.1$ sq. ins. and two sides are $2.8''$ and $3''$. Measure the third side.

21. On a base of 5.5 cm. construct a Δ of area 9.9 sq. cm., one of its angles being 65° . Measure the vertical angle.

22. Draw a ΔABC on a base $1.7''$ long, having an area of 1.275 sq. ins. and the median bisecting the base being $1.9''$ long. Measure the other sides.

23. Construct a parallelogram whose diagonals are $3.5''$ and $3''$ and whose area is 4.2 sq. ins. Measure the acute $\angle$ between the diagonals.

24. A level crossing is made by a road 20 ft. wide crossing a railway 25 ft. wide at an angle of 42° . Draw a figure to scale and so find the area of the crossing.

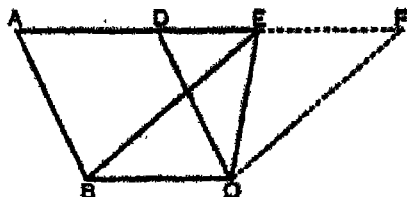
25. $ABCD$ is a quadrilateral the area of which is bisected by the diagonal AC . Draw the figure, given that $AB = 9.6$ cm., $BC = 4.2$ cm., $AC = 8.8$ cm., $\angle BCD = 128^\circ$. Measure CD .

THEOREM 28.

If a parallelogram and a triangle stand on the same base and between the same parallels, the area of the parallelogram is double that of the triangle.

Given a par^m ABCD and a Δ EBC on the same base BC and between the same $\parallel$ s BC and AE.

To prove that par^m ABCD is twice the Δ EBC in area.



Constr. Draw $CF \parallel BE$, completing the par^m EBCF.

Proof. EC is a diag. of the par^m BF,

$\therefore \text{par}^m$ BF = twice Δ EBC in area; (*diag. bisects par^m*)

but par^m BF = par^m AC, (*same base BC and same $\parallel$ s*)

$\therefore$ area of par^m AC = twice area of Δ EBC.

Q.E.D.

EXERCISES ON PARALLELOGRAMS AND TRIANGLES.

EXAMPLES 44.

THEORETICAL.

1. Prove that, if a parallelogram and a triangle are between the same parallels, and the base of the triangle is twice the base of the other, the figures are equal in area.

2. If a par^m and a Δ are on the same base and the par^m is twice the Δ in area, what is known about their altitudes?

3. If a par^m and a Δ are equal in area and are on the same base, what can be said about their altitudes?

4. P is any point inside a par^m ABCD. Prove that

$$\Delta PAD + \Delta PBC = \text{half } \text{par}^m \text{ ABCD.}$$

5. ABCD is a parallelogram and PQRS is a quadrilateral whose vertices are on the sides of ABCD and such that PR is parallel to a side of ABCD. Prove that $ABCD = \text{twice } PQRS$ in area.

6. ABCD is a trapezium in which $AB \parallel CD$. If X bisects BC, prove that $\Delta XAD = \text{half the trapezium in area}$.

PRACTICAL.

7. Draw a square on a base of 1.5"; make an isosceles Δ on a base of 1.5" equal to the square. Measure its angles.

8. Construct a par^m ABCD in which $AD = 1.2''$, $AB = 1.8''$, and $\angle A = 50^\circ$; on AD describe a right-angled Δ equal in area to ABCD. Measure the hypotenuse.

AREAS OF OTHER RECTILINEAR FIGURES.

I. THE TRAPEZIUM.

In the trapezium ABCD, the diagonal BD is joined. The figure is now divided into two Δ s BCD, ABD which have equal heights BX, DY as shown. (Why?)

Let $BX = DY = h$.

$$\begin{aligned}\therefore \text{Area of trapezium} &= \Delta BCD + \Delta ABD \\ &= \frac{1}{2} DC \cdot h + \frac{1}{2} AB \cdot h \\ &= \frac{1}{2} (DC + AB) h,\end{aligned}$$

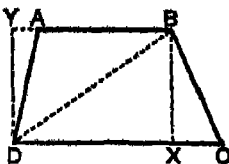


FIG. 145.

$\therefore$ the area of a trapezium
= half the sum of the parallel sides $\times$ the perp^r distance between them.

II. THE QUADRILATERAL.

The area of a quadrilateral can be found in several ways, in addition to the method of counting squares:

(a) It can be reduced to an equivalent triangle by Problem 6 and the area of that triangle can then be found.

(b) It can be divided into two triangles by drawing a diagonal and the area of each triangle calculated from any base and the corresponding height. In particular, the diagonal can be taken as a common base.

We thus get the figure drawn, showing the perp^a AX, CY drawn from A and C to the diag. BD.

[These perp^a are called the offsets.]

$$\begin{aligned}\text{Quad^r ABCD} &= \Delta BCD + \Delta ABD \\ &= \frac{1}{2} BD \cdot CY + \frac{1}{2} BD \cdot AX \\ &= \frac{1}{2} BD (CY + AX)\end{aligned}$$

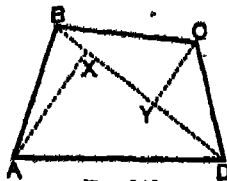


FIG. 146.

$\therefore$ the area of a quadrilateral
= half a diagonal $\times$ the sum of its offsets.

III. THE POLYGON.

The area of a polygon can be found in several ways, in addition to the method of counting squares;

(a) It can be reduced to an equivalent triangle by repeated use of the method of Problem 8, and the area of that triangle can then be calculated;

(b) The polygon can be divided into triangles by joining *corner to corner*, and its area found by adding up the areas of the separate triangles;

(c) The polygon can be divided into *right-angled triangles and trapeziums* by the Field-Book method. This is the principle which surveyors use in finding the area of a field bounded by straight lines. An illustration will show the method. The field-book entry is read upwards and the measurements are all taken from the bottom (D) of the long line (AD). Thus DS = 32 yds., and DR = 40 yds.; $\therefore$ RS = 8 yds.

The lengths of the offsets are put to the right or left of the numbers in the central column. The numbers have been put in the figure direct from the field-book.

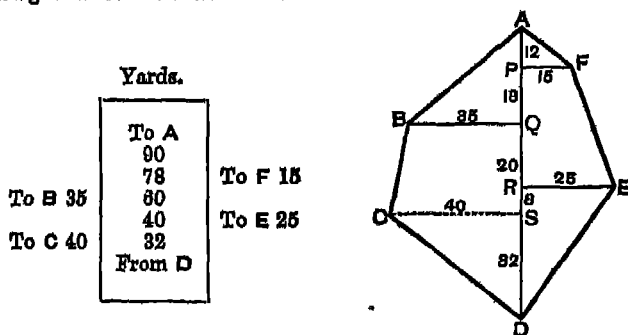


Fig. 147.

Thus:

Area of $\triangle APF = \frac{1}{2} AP \times PF = \frac{1}{2} \times 12 \times 15$	sq. yds. = 90
Area of trap. PE = $\frac{1}{2} (PF + RE) \times PR = \frac{1}{2} \times 40 \times 38$	= 760
Area of $\triangle DRE = \frac{1}{2} DR \times RE = \frac{1}{2} \times 40 \times 25$	= 500
Area of $\triangle DSC = \frac{1}{2} DS \times SC = \frac{1}{2} \times 32 \times 40$	= 640
Area of trap. BS = $\frac{1}{2} (BQ + CS) \times QS = \frac{1}{2} \times 75 \times 28$	= 1050
Area of $\triangle ABQ = \frac{1}{2} AQ \times BQ = \frac{1}{2} \times 80 \times 35$	= 525

Total area	...	...	= 3565
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EXERCISES ON AREAS OF QUADRILATERALS AND POLYGONS.

EXAMPLES 45.

ARITHMETICAL AND THEORETICAL.

1. The parallel sides of a trapezium are $6\cdot2''$ and $2\cdot2''$ and the perp^r distance between them is $5\cdot5''$. Calculate its area.

2. The area of a trapezium is 6 sq. ins. and the parallel sides are $4''$ and $2\cdot4''$ long. Find the perpendicular distance between these sides.

3. ABCD is a quadrilateral, having the diagonal AC $10\cdot8$ cm., and the offsets BE and DF from AC being $5\cdot6$ cm. and $6\cdot6$ cm. long respectively. Calculate the area of ABCD.

4. Show by means of figures that many quadrilaterals can be drawn with a given diagonal and with given offsets from it. Why are all their areas equal?

5. Show that two congruent trapeziums can be so placed as to form a parallelogram. Hence deduce the area of a trapezium.

6. Prove that the area of a quadrilateral, whose diagonals cut at right angles, is half the product of the diagonals.

7. ABCD is a quadrilateral, and P moves so that the areas of the quadrilaterals ABCD, ABCE are equal. Find the locus of P.

8. Draw plans of the polygons referred to in these field-book entries. Calculate their areas.

(i)

	Yards.
To C	80
To B 9	72
	55
	21
From A	

(ii)

	Yards.
To A	40
To B 6	32
	18
	12
	4
From C	

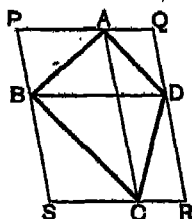


FIG. 148.

9. Calculate the area of a kite whose diagonals are $3''$ and $4''$.

10. AB, CD, EF are parallel lines, met by two transversals ACE and BDF. Prove that the quadrilaterals BCED, ACFD are equivalent.

PRACTICAL.

12. Draw a quadrilateral $ABCD$ such that $AB = 2''$, $BC = 2.4''$, $CD = 1.8''$, $DA = 2.5''$, and diagonal $AC = 3''$. By drawing and measuring offsets from AC , calculate the area.

13. Draw a kite whose sides are 6 cm., 6 cm., 8 cm., 8 cm., the angle between the two last sides being 40° . Find its area.

14. An isosceles trapezium has one of its parallel sides 3" long, the oblique sides .8" each, and the angle they make with the 3 inch side is 60° . Measure the distance between the parallel sides and find the area of the trapezium.

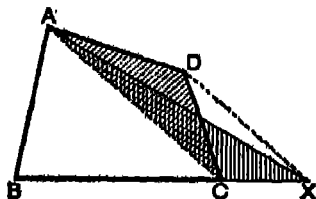
15. $ABCD$ is a four-sided field in which $\angle D = 75^\circ$, $\angle C = 120^\circ$, $\angle B = 90^\circ$, $CD = 400$ yds., $BC = 240$ yds. Draw the field to scale and use the figure to find the area of the field itself.

PROBLEM 6.

To construct a triangle equal in area to a given quadrilateral.

Given a quad^l $ABCD$.

Required to construct a Δ equal in area to $ABCD$.



Constr.

Join AC ;

draw $DX \parallel AC$ to meet BC produced in X ; join AX .

Then ΔABX shall be equal in area to quad^l $ABCD$.

Proof. $\Delta ACX = \Delta ACD$. (same base AC and same $\parallel$ s)

To each add ΔABC ,

$\therefore \Delta ABX = \text{quad}^l ABCD$ in area.

Q.E.F.

Cor. 1. *To construct a triangle equal in area to a given pentagon (or polygon).*

Let $ABCDE$ be the given pentagon.

Join AD and draw $EX \parallel AD$, to meet CD produced in X .

Join AC and draw $BY \parallel AC$, to meet DC produced in Y .

Join AX, AY .

Then $\triangle AYX$ is the $\triangle$ required.

It can now be easily shown, by the method of proof used in Problem 6, that pentagon $ABCDE = \text{quad}^l ABCX$

$= \triangle AXY$ in area.

Q.E.F.

So too with a polygon of more than five sides, cutting off one corner at a time.

Cor. 2. *To construct on a given base a triangle equal in area to a given triangle.*

Let BCD be the given triangle, and BA the given base (laid along the side BD).

This is only a particular case of Problem 6, the quad^l $BADC$ being flattened down at A to form the $\triangle B(A)DC$. Hence:

Join AC , and draw $DX \parallel AC$ to meet BC produced in X .

Join AX .

Then $\triangle ABX$ is the $\triangle$ required.

The proof is as in Problem 6.

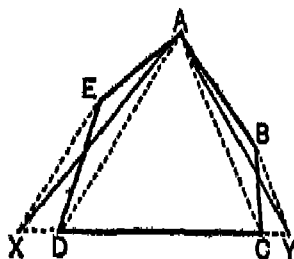


FIG. 149.

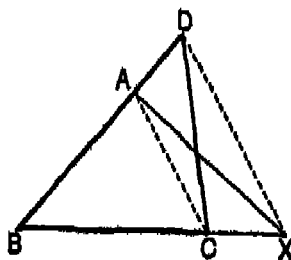


FIG. 150.

REDUCING TO AN EQUIVALENT TRIANGLE.

EXAMPLES 46. PRACTICAL.

1. ABCD is a quadrilateral in which $AB = 1.6''$, $BC = 1.8''$, $CD = .8''$, $DA = 2.4''$, $\angle D = 60^\circ$. Draw the figure and reduce it to an equal triangle. Hence calculate the area of the quadrilateral.

2. Draw a kite whose sides are 8 cm., 6 cm., 10 cm., 10 cm., the angles between the unequal sides being 130° . By constructing an equivalent triangle, calculate the area of the kite.

3. Draw a quadrilateral whose sides are $1.5''$, $2''$, $2.5''$, $3''$ in order, the diagonal joining the intersections of the first pair and the second pair of sides being $2.4''$. Draw a triangle equal to it in area and having a side $2.5''$ long, hence find the area of the figure.

4. Inside a circle of radius $2''$ draw a regular pentagon. (Note what will be the size of the angle subtended at the centre of the circle by each side of the pentagon.) Reduce the pentagon to an equivalent triangle, and hence calculate the area of the pentagon.

5. Find the area of the quadrilateral ABCD, if $BC = 5$ cm., $AD = 13$ cm., $A = 90^\circ$, $B = 135^\circ$, $C = 90^\circ$.

6. Describe an equilateral triangle (side $1''$) and outside it a square (side $1''$), the two figures having one side common. Construct a triangle equal in area to the complete figure.

TO MAKE A PARALLELOGRAM EQUAL TO A TRIANGLE.

EXAMPLES 47.

1. Prove that a rectangle is equal in area to a triangle if its altitude is half the altitude of the triangle, and the figures stand on equal bases (see Fig. 152).

2. Use Ex. 1 to explain how to draw a rectangle equal in area to a triangle, the two figures standing on the same base.

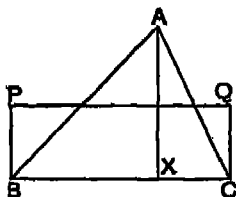


Fig. 152.

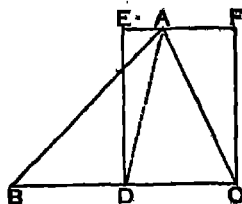


Fig. 153.

3. Prove that a rectangle and a triangle are equal in area if they have equal altitudes and the base of the triangle is twice the base of the rectangle (see Fig. 153).

4. Explain how to draw a rectangle equal in area to a triangle and on a base half the base of the triangle.

[In Fig. 153 $\triangle ABD = \triangle ADC$ (equal bases and same alt.)

$$\therefore \triangle ABC = 2 \triangle ADC;$$

But rect. CDEF = 2 $\triangle ADC$ (same base and same alt.)

$$\therefore \triangle ABC = \text{rect. CDEF.}]$$

5. Explain how to alter the constructions in Nos. 2 and 4, so as to enable a parallelogram to be drawn equivalent to a given triangle and having an angle equal to a given angle.

PRACTICAL EXAMPLES.

6. Draw a $\triangle ABC$ such that $AB = 4$ cm., $BC = 5$ cm., $CA = 6.1$ cm. Draw a rectangle on a base of 6.1 cm. equal to the triangle in area. (See Ex. 2.)

7. Draw a $\triangle$ whose sides are 2", 2.3" 1.8". Draw a rectangle equal in area to the $\triangle$, but on a base of 1". Measure its height.

8. Draw a $\triangle ABC$ such that $BC = 7$ cm., $\angle B = 47^\circ$, $\angle C = 80^\circ$. Draw an equivalent parallelogram on a base of 3.5 cm. with one angle of 65° . Measure the adjacent sides.

9. Draw a $\triangle ABC$ whose sides are 5.4 cm., 9 cm., 8.2 cm. Construct an equal rhombus whose sides are 8.2 cm. Measure one of its acute angles.

The above constructions contain two methods by which a parallelogram or rectangle can be drawn equal to a given triangle. The methods depend on drawing—

(i) the *par^m* on the same base and with half the altitude of the triangle, or

(ii) the *par^m* on half the base and with the same altitude as the triangle.

THEOREM OF PYTHAGORAS.

We have already seen (see page 48) that, in certain right-angled triangles, the square on the hypotenuse is equal to the sum of the squares on the other two sides.

Let us examine other cases to see whether this fact is always true.

In Fig. 156 $\triangle ABC$ is right-angled at B and the figure is completed to show the square on AC enclosed inside another square. If $BC = 4''$ and $AB = 5''$, and the three other triangles are congruent with ABC , calculate the area of each triangle and the area of the smaller square. Is the square on AC equal to the sum of the squares on AB and BC ?

Examine Fig. 156 again. Find whether $AC^2 = AB^2 + BC^2$

- (i) if $AB = 9''$, $BC = 3''$;
- (ii) if AB and BC have any lengths you please;
- (iii) if $AB = a$, $BC = b$.

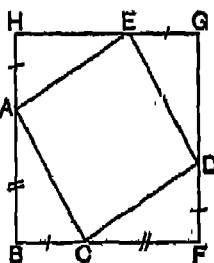


Fig. 156.

We can see the truth of the same fact by means of our eyes in several ways.

I. PQRS and XQAC are two squares placed side by side as in figure, so that QRA is a straight line.

Mark off a point B between Q and R, so that $QB = CA$.

Join PB and BC. Cut out $\triangle PQB$ and place it above PS in the position PSD. Cut out $\triangle ABC$ and show that it exactly fills up the space DXC. Convince yourself that PBQD is a square.

As it is a square made up of the two original squares and PQRS is the square on QR, or BA, it follows that

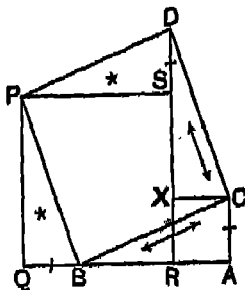


Fig. 157.

$$BC^2 = AB^2 + AC^2.$$

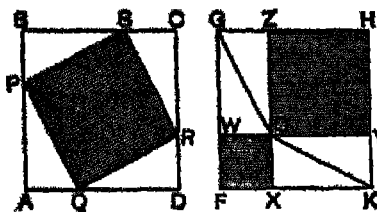


FIG. 168.

II. $ABCD$ and $FGHK$ in Fig. 168 are equal squares.

In $ABCD$ four equal lengths are marked off, BP , AQ , DR , CR , and in $FGHK$ two more of the same length, FW , FX .

If the figure $PQRS$ is joined up, it will be a square.

If in the other figure lines WO , ZO are drawn parallel to the sides, WX and ZY will be squares.

Also the four triangles in the first figure can be refitted into the unshaded rectangles in the second figure

$\therefore$ the shaded parts in the two figures are equal.

Also $QD = ZH$ and $RD = FW$,

$\therefore QR^2 = QD^2 + RD^2$ in the right-angled $\triangle QRD$.

HISTORICAL NOTE

The Theorem of Pythagoras obtains its name from the famous Greek mathematician who died in 500 B.C. It is not certain how he proved the theorem, but it is probable either that his proof is based on method II shown above, or that it is the one given as Corollary 3 to Theorem 61.

The proof on page 127 is due to Euclid, of Alexandria, who lived from about 330 B.C. to about 275 B.C. Though it is long, each part of it is quite simple, and the pupil should try to make it up for himself from the following headings.

[We try to prove that rect. $AQ = sq.$ AD , and similarly that rect. $PF = sq.$ CK . Would this prove what is required?

But rect. $AQ = \text{twice } \triangle BAG$ (why?)

and sq. $AD = \text{twice } \triangle EAO$ (if DEO is straight; is it?).

We $\therefore$ have to prove that $\triangle s$ BAG , EAO are $\cong$ or $\equiv$.

But they are $\equiv$, if only $\angle EAO = \angle BAG$. Is this so?]

Hence we prove that:—

- (i) DBC is a straight line;
- (ii) $\angle EAO = \angle BAG$;
- (iii) $\triangle EAO \equiv \triangle BAG$;
- (iv) sq. AD and rect. AQ are double the above $\triangle s$;
- (v) sq. $AD = \text{rect. } AQ$;
- (vi) sq. $CK = \text{rect. } CQ$ similarly;
- (vii) sq. AD and $CK = sq.$ AF .

THEOREM 28 (Theorem of Pythagoras).

The square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides.

Given a $\triangle ABC$ with $\angle B$ a right angle.

To prove that

sq. on AC = sq. on AB + sq. on BC .

Constr. Make the sqs on AC , AB , BC , viz. $AQFG$, $ABDE$, $BOHK$; join EQ and BG .

Draw through B a line $BPQ \parallel AG$.

Proof. Since $\angle$ s DBA and ABC are right $\angle$ s,

$\therefore DBC$ is a straight line.

Again rt. $\angle EAB =$ rt. $\angle CAG$;

adding $\angle BAC$ to each, we get
 $\angle EAC = \angle BAG$.

Hence in $\triangle$ s EAC , BAG

$$\begin{cases} EA = BA, & \text{(sides of a sq.)} \\ AC = AG, & \text{(sides of a sq.)} \\ \angle EAC = \angle BAG, & \text{(proved)} \end{cases}$$

$\therefore \triangle$ s $\equiv$, and equal in area. (two sides & incl. $\angle$)

But sq. AD is twice $\triangle EAC$. (same base EA and same h)
and rect. AQ is twice $\triangle BAG$, (same base AG and same h)

$\therefore$ sq. $AD =$ rect. AQ .

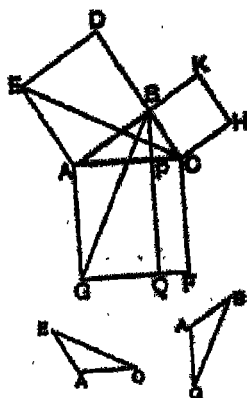
Similarly it can be shown, by joining AH and BF , that

sq. $CK =$ rect. CQ .

$\therefore$ sq. $AD +$ sq. $CK =$ rect. $AQ +$ rect. CQ
 $=$ sq. AF ,

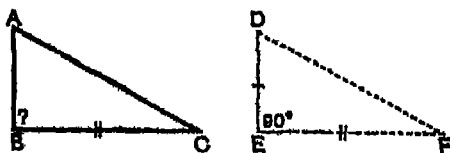
that is, the square on $AC =$ sum of the squares on AB , BC .

Q.E.D.



THEOREM 80 (Converse of Pythagoras).

If the square on one side of a triangle is equal to the sum of the squares on the other two sides, the angle contained by these two sides is a right angle.



Given a $\triangle ABC$, in which $AC^2 = AB^2 + BC^2$.

To prove that $\angle B$ is a right angle.

Constr. Draw a $\triangle DEF$ in which $DE = AB$ and $EF = BC$, and the $\angle DEF$ is a rt. $\angle$.

Proof. In $\triangle ABC$, $AB^2 + BC^2 = AC^2$, (given)

and in $\triangle DEF$, $DE^2 + EF^2 = DF^2$, ($\angle E$ a right $\angle$)

but $AB = DE$ and $BC = EF$, (constr.)

$$\therefore AC^2 = DF^2,$$

$$\therefore AC = DF.$$

Now in $\triangle s ABC, DEF$,

$$\begin{cases} AB = DE, & (\text{constr.}) \\ BC = EF, & (\text{constr.}) \\ \text{and } AC = DF, & (\text{proved}) \end{cases}$$

$$\therefore \triangle s \equiv; \quad (\text{three sides})$$

so that $\angle B = \angle E$.

$\angle E$ was made a right $\angle$,

$\therefore \angle B$ is a right $\angle$.

Q.E.D.

EXERCISES ON PYTHAGORAS AND ITS CONVERSE.

EXAMPLES 48.

ARITHMETICAL.

1. ABC is a triangle in which A is a right angle.
 - (i) If $AB = 5''$, $BC = 13''$, calculate AC.
 - (ii) If $AB = 9$ cm., $AC = 12$ cm., find BC.
 - (iii) If $AB = 2.5''$, $AC = 6''$, find BC.
2. Calculate the hypotenuse of a right-angled triangle if the other sides are (i) 15 ft. and 36 ft.; (ii) 48 ft. and 14 ft.
3. Calculate the third side of a right-angled triangle
 - (i) if the hypotenuse is 76 cm. and another side is 72 cm.;
 - (ii) if the longest side is 85 ft. and another side is 84 ft.
4. A rectangle measures 20 ft. by 5 ft. 10 ins. Find its diagonal.
5. A ship sails 48 miles East and then 64 miles North. How far is it finally from its starting point?
6. The diagonal of a rectangle is 12.2 cm. and one side is 12 cm. Find the adjacent side.
7. A ladder just reaches a window-sill 63 ft. above the ground and its base is 16 ft. from the foot of the wall below the window. Find the length of the ladder.
8. Which of the triangles whose sides are given below are right-angled?
 - (i) $5''$, $6''$, $7''$; (ii) $12''$, $35''$, $37''$; (iii) 2 cm., 4.8 cm., 5.2 cm.
9. A ladder 65 ft. long can reach a height of 52 ft. up a wall on one side of a street and when its top is turned right over it can reach a height of 25 ft. on the other side of the street. Calculate the width of the street.
10. Find the length of a wireless aerial which joins the tops of two masts 35 and 42 feet in height, their distance apart being 20 ft. (Answer to nearest tenth of a foot.)
11. ABC is a triangle and AD an altitude. If $BD = 3''$, $DC = 12''$, $AD = 6''$, prove that $\angle BAC = 90^\circ$.
12. The non-parallel sides of a trapezium are $15''$ and $13''$, the $\perp$ distance between the $\parallel$ sides being $12''$ and the shorter of them $3''$. Find the fourth side and the area.
13. ABCD is a quadrilateral and $\angle C = 90^\circ$. If $BC = 3$ ft., $CD = 4$ ft., $AD = 11$ ins., $AB = 5$ ft. 1 in., prove that the angle between AD and the diagonal BD is 90° .
14. A room is $22' 6''$ long, $18'$ wide, and $13' 6''$ high; find the lengths of the diagonals of (i) each wall; (ii) the floor.

(SHORTER.)

K

15. If a box measures 12" by 3" by 4", find the length of the diagonal joining one corner to the opposite corner.

16. In a cone

(i) slant side = 15"; diameter of base = 2'; find the height;

(ii) height = 9", slant side = 41"; find the radius of the base;

(iii) slant side 10 cm., altitude = $\sqrt{84}$ cm.; find the diameter of the base.

17. A square pyramid is 12 ins. in height and the edge of its base (a square) is 18 ins. Find the area of each of the triangular faces.

18. A right pyramid (height 36") stands on a rectangular base measuring 50" $\times$ 96". Calculate the slant sides and the areas of the four slant faces.

19. In a $\triangle ABC$, $\angle A = 90^\circ$, $b = 3''$, $c = 4''$; find the area and the altitude AX .

20. As in Ex. 19 calculate the perpendicular to the hypotenuse from the right-angle of a right-angled triangle given that the sides about that angle are 7 ins. and 24 ins.

THEORETICAL.

21. Show that the square drawn on the diagonal of a square is twice the area of the square on one of its sides.

22. If two squares are given, show how to find a line the square on which is equal to (i) the sum, (ii) the difference of the areas of the given squares.

23. ABC is an equilateral triangle and $AD \perp BC$; prove that $AD^2 = 3 BD^2$.

24. If AX is an altitude of a $\triangle ABC$, prove that $BX^2 + AC^2 = CX^2 + AB^2$.

25. ABC is a $\triangle$ in which $\angle A = 90^\circ$; P is a point on AB and Q on AC . Show that $CP^2 + BQ^2 = PQ^2 + BC^2$.

26. If CX and BY are medians of the $\triangle ABC$ which has a right-angle at A , prove that $BY^2 + CX^2 = 5 XY^2$.

27. If X is any point in the altitude AD of a $\triangle ABC$, and $AB > AC$, show that $AB^2 - AC^2 = XB^2 - XC^2$.

28. If $ABCD$ be a rhombus, prove that $AC^2 + BD^2 = 4 AB^2$.

29. Prove that the sum of the squares on the sides of a rectangle is equal to the sum of the squares on its diagonals.

30. If P is any point inside a rectangle $ABCD$, prove that $PA^2 + PC^2 = PB^2 + PD^2$.

31. In the figure to Theorem 29, prove that

(i) $AO \times BP = AB \times BC$;

(ii) $AB^2 = AP \times AC$.

32. ABC is an equilateral triangle and D is a point in BC such that $BD = \frac{1}{3} BC$; prove that $AD^2 = \frac{7}{9} BC^2$. (Let $BC = 3a$.)

33. O is a point inside ΔABC , and OX, OY, OZ are drawn perpendicular to BC, CA, AB . Prove that

$$BX^2 + CY^2 + AZ^2 = XC^2 + YA^2 + ZB^2.$$

34. ABC is an equilateral Δ and D is a point in BC such that $BD = \frac{1}{4} BC$. Show that $AD^2 = BD^2 + 4 DC^2$.

PRACTICAL CONSTRUCTIONS BASED ON PYTHAGORAS' THEOREM.

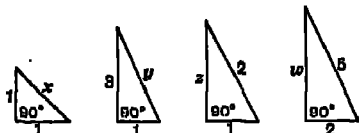


FIG. 159.

EXAMPLES 49.

1. Calculate x and y in Fig. 159 in the form $\sqrt{\quad}$. How can lines $\sqrt{2}$ '' long be drawn with ruler and compasses?

2. Calculate z and w above in the form $\sqrt{\quad}$. How can lines $\sqrt{8}$ cm., $\sqrt{21}$ cm. long be drawn with ruler and compasses?

3. Using Pythagoras, and noting that $2^2 + 1^2 = 5$, $3^2 - 2^2 = 5$, draw a line $\sqrt{5}$ '' long

- (i) making it the hypotenuse of a right-angled Δ ;
- (ii) making the hypotenuse 3'' long.

4. Being given a line $\sqrt{10}$ '' long, explain how to use it to draw a line $\sqrt{11}$ '' long.

5. Draw a line $\sqrt{5}$ '' long and use it to draw a line $\sqrt{6}$ '' long.

6. Without extracting the square root, draw lines of lengths—

- (i) $\sqrt{10}$ ''; (ii) $\sqrt{13}$ ''; (iii) $\sqrt{21}$ cm.; (iv) $\sqrt{24}$ cm.

Measure each and check your construction by working out the square roots to 2 decimal places.

7. Draw a line AB 1.1 inches long. Make lines X, Y whose squares are respectively (i) twice, (ii) three times as large as the square on AB .

8. Draw lines whose lengths are

- (i) $\sqrt{2 \cdot 7^2 + 1 \cdot 1^2}$ inches; (ii) $\sqrt{2 \cdot 3^2 - 1 \cdot 4^2}$ inches.

Measure each in inches correct to 2 decimal places.

SECTION III.

LOCI.

Elementary Ideas.

Example 1.—A boy hides an article in a garden bounded by a straight wall ABC and an irregular hedge. B is the doorway into the garden. The article is hidden under the ground at P. Later on the boy wishes to find the point P at which he must begin to dig. How can he do this?

Suppose he had noticed that PB is 14 ft. and that PD, the perpendicular from P to AC, is 10 ft. When he comes back again he asks himself—

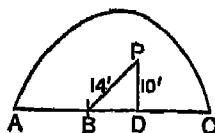


Fig. 161.

- (i) where are all the points in the garden which are 14 ft. away from B. The answer is of course—

A part of a circle (XYZ) drawn with centre B and with a radius of 14 ft. (see Fig. 162).

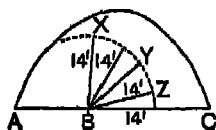


Fig. 162.

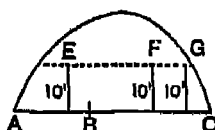


Fig. 163.

- (ii) where are all the points in the garden which are 10 ft. away from the wall AC. The answer is—

A part of a line (EFG) parallel to AC and 10 ft. away from it (see Fig. 163).

The point P must be where the circle PQR and the line EFG meet.

Each of these lines, XYZ and EFG, is an illustration of a locus.

Definition. The locus of a point which fulfils a certain condition is the *path* it traces out as it moves in accordance with that condition.

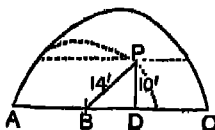


Fig. 164.

Referring to the above Example, we see that

- (i) when the condition P has to fulfil is that $BP = 14$ ft., the locus of P (the path it traces out) is the arc XYZ of a circle;

- (ii) when the condition P has to fulfil is that PD , the perpendicular to BC , is 10 feet, the locus of P is the line EFG parallel to ABC .

Ex. 2.—What is the locus of a point P which moves in a plane so that it is always 1 in. away from a fixed point O ?

Ex. 3.—What is the locus of a donkey tethered to a peg by a taut rope?

Ex. 4.—What is the locus of the centre of a boy's hoop as he rolls the hoop straight along the ground?

Ex. 5.—What is the locus of my right foot if I "*stand still*" in a lift while the lift is ascending?

Ex. 6.—State the locus of a point which moves on a sheet of paper so as to be

- (i) 2" from the bottom of the page;
- (ii) equidistant from the top and bottom of the page;
- (iii) equidistant from the top and the right-hand edge of the page;
- (iv) one inch from a line drawn right across the page. (Is there more than one part of the locus in this case?)

Ex. 7.—Where are all the points on the earth's surface (considered as a sphere) which are 100 miles north of the Equator?

Ex. 8.—What is the locus of all points on the earth's surface whose Latitude is $66\frac{1}{2}^\circ$ north?

Later on we shall need to discuss the locus of a point by reasoning, but at present the pupil should try to get used to the idea of a locus, and to see for himself what sort of a path the moving point seems to trace out. A careful drawing, even a rough drawing at times, is useful. The figure is often a straight line or a circle, but it may take other shapes also. Sometimes the locus may be only a portion of a curve, such as an arc of a circle.

EASY PRACTICAL EXERCISES ON LOCI.

EXAMPLES 50.

Many of the harder questions following can be discussed with advantage by teacher and class together; those marked with an asterisk can be treated from the point of view of Solid Geometry, as well as in the ordinary way by the methods of Plane Geometry.

*1. Find the locus of all points which are 1" away from a given point O .

*2. Find the locus of all points which are 1" away from a given straight line AB .

*3. Mark two points A and B on a sheet of paper, AB being 2". Mark 6 points at least, each one of which is as far from A as it is from B. What do you think is the locus of points which are equidistant from A and B?

How does the locus lie with reference to AB in direction, and how does it lie in relation to the middle point of AB?

4. A sheep is at one side of a field and it needs to go to the opposite side, but two sheep dogs, fastened to stakes in the ground by short ropes, and equally distant from the sheep, are in the way. If the sheep naturally walks so as to be never nearer to one dog than it is to the other, what is the path that the sheep will take across the field? [Ignore the lengths of the ropes.]

5. A pole BC 20 ft. long slides along the ground with its ends on two horizontal rods at right angles. Draw a figure to scale, showing that the locus of P, the middle point of the rod, is an arc of a circle.

6. A set-square ABC moves so that the longest edge runs along a fixed straight line on your paper. Mark several positions of the vertex A. What is the locus of A?



Fig. 165.

7. A set-square ABC moves so that one corner A (not the right angle) is fixed. What is the locus of B? of C? of any point between B and C?

8. A set-square ABC ($\angle B = 90^\circ$) moves so that AB always passes through a fixed point X and BC through a fixed point Y. Taking $XY = 2''$, and moving your set-square about on paper, find the locus of B. [Put pins in your paper to fix the positions of X and Y.] (See Fig. 166.)

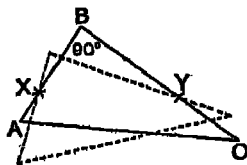


Fig. 166.

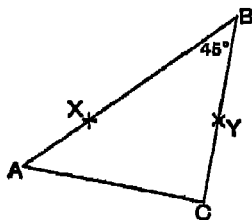


Fig. 167.

9. Do Ex. 8 with a 45° set-square, taking B to be 45° (Fig. 167).

10. AB and AC are two straight lines representing a road and a railway perpendicular to the road. By plotting points find the locus of a man P who walks so as to be always as far from the road as he is from the railway.

11. An aeroplane X is flying so that it is always just over Y , some point or other in a straight road AB , and it is found that the height XY of the aeroplane is always twice the distance of Y from the town A . Plot the locus of the aeroplane.

12. What is the locus of a speck of paint on a door as the door turns on its hinges? What is the locus of the tip of the door knob?

13. A grandfather's clock is carefully tilted about the front line of the rectangle which forms the stand on the floor. What is the locus (i) of the centre of the dial; (ii) of a particular minute mark on the dial; (iii) of any definite mark on the side of the clock-case?

*14. What is the locus of centre P of a circle of radius $1''$, the circumference of which always goes through a fixed point O ?

15. A child holds an iron hoop horizontally so that one point of the hoop is caught by a nail on the ground. The nail is 2 ft. from each of two perpendicular edges of the floor of a room. If the hoop has a radius of 1 ft. 6 ins., what is the locus of the centre of the hoop as the child takes up all possible positions with it?

16. A man wheels a water-tub to which is attached a hand-spray which can water the ground up to a distance of 10 ft. from him. If he walks along the outer boundary of a rectangular lawn, find the boundary of

- (i) the portion of the lawn that can be watered;
- (ii) the portion of the garden (surrounding the lawn) that can be watered.

17. Draw a triangle with sides all longer than $2''$. Trace the locus of a point P which moves so that it is one inch away from one side of the triangle and never less than one inch away from each of the other sides.

18. ABC is an equilateral triangle of side $2''$. Find the locus of a point P which moves so as to be $1''$ away from one of the corners A , B , C , and never less than $1''$ away from each of the other corners.

19. A semi-circular protractor slides in its own plane with the ends of the base always on two fixed straight lines OA , OB at right angles to each other. Find by drawing the locus of any mark on the rim (say the 40° mark).

LOCI CONSIDERED THEORETICALLY.

Although it is a very good thing to be able to see, either in your mind's eye or actually on paper, what is the locus of a point in a particular problem, you have not completed the investigation until a proof has been given; in fact, two proofs are generally necessary.

Thus in finding the locus of a point P equidistant from two fixed points A and B , we see that one position of P is evidently at the middle point of the straight line AB ; other positions of P are such as those marked on the figure.

These *suggest* that the locus is a straight line, which :

- (i) goes through C the middle point of AB ,
- and (ii) is perpendicular to AB .

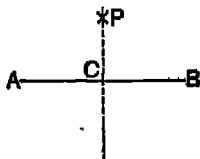


FIG. 168.

We have now to prove two things:—

- (a) that, if P is equidistant from A and B , it lies on the perpendicular bisector of AB . To do this, join P to C (the middle point of AB) and prove by congruent triangles that $\angle PCA$ and $\angle PCB$ are equal, and $\therefore$ are right angles;
- (b) that, if P is on the perpendicular bisector of AB , then $PA = PB$.

This also is a case of congruent triangles.

THEOREM 31.

The locus of a point equidistant from two given points is the perpendicular bisector of the straight line joining the two given points.

I. Given two fixed points A and B, and P, any point such that $PA = PB$.

Required to prove that P lies on the perp^r bisector of AB.

Construction. Bisect AB at C and join PC.

Proof.

In Δs PCA, PCB,

$$\begin{cases} PA = PB, & (\text{given}) \\ AC = CB, & (\text{constr.}) \\ \text{and PC is common,} & \\ \therefore \Delta s \equiv, & (3 \text{ sides}) \end{cases}$$

so that $\angle PCA = \angle PCB$,

$\therefore$ PCA and PCB are rt. $\angle s$, (= adj. $\angle s$)

and PC is perp^r to AB; (definition)

that is, P lies on the perp^r bisector of AB.

II. Given that P is any point on the perp^r bisector of AB.

Required to prove that $PA = PB$.

Proof.

In Δs PCA, PCB,

$$\begin{cases} AC = CB, & (\text{given}) \\ PC \text{ is common,} & \\ \text{and } \angle PCA = \angle PCB, & (\text{right } \angle s) \\ \therefore \Delta s \equiv, & (2 \text{ sides and incl. } \angle) \end{cases}$$

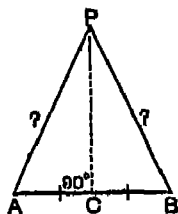
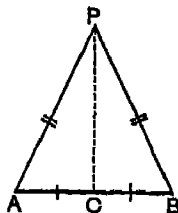
so that $PA = PB$.

Hence by I. every point P which is equidistant from A and B is on the perpendicular bisector of AB,

and by II. every point P on the perpendicular bisector of AB is equidistant from A to B,

$\therefore$ the locus of points equidistant from A and B is the perpendicular bisector of AB.

Q.E.D.



THEOREM 32.

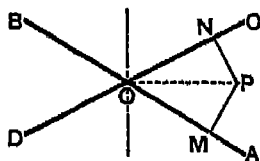
The locus of a point equidistant from two given straight lines is the pair of lines which bisect the angles between the two given lines.

I. Given two straight lines AB and CD meeting at O, and P, any point (inside $\angle AOC$) that is equidistant from AB and CD, so that perp^m PM, PN drawn to AO, CO respectively are equal.

Required to prove that P lies on the bisector of $\angle AOC$.

Construction. Join PO.

Proof. In Δ s OPM, OPN,
 $\left\{ \begin{array}{l} \angle \text{s M and N are rt. } \angle \text{s, (constr.)} \\ \text{OP is common,} \\ \text{and PM = PN, (given)} \end{array} \right.$
 $\therefore \Delta \text{s} \equiv$, (rt. $\angle$, hyp., 1 side)
 so that $\angle \text{POM} = \angle \text{PON}$,



that is, P lies on the bisector of $\angle AOC$.

II. Given PQ and RS, the bisectors of the angles between the given lines AB and CD, and take P, a point on one of the bisectors.

Required to prove that the perp^m PM, PN are equal.

Construction. Join PO.

Proof.

In Δ s OPM, OPN,
 $\left\{ \begin{array}{l} \angle \text{POM} = \angle \text{PON, (given)} \\ \angle \text{M} = \angle \text{N, (right } \angle \text{s)} \\ \text{and OP is common,} \end{array} \right.$
 $\therefore \Delta \text{s} \equiv$, (2 $\angle$ s and a side)
 so that PM = PN.

Hence by I. every point P which is equidistant from AB and CD is on one of the bisectors of the angles between the given lines;

and by II. every point P which is on one of the bisectors of the angles is equidistant from AB and CD.

$\therefore$ the locus of points equidistant from AB and CD is the pair of bisectors of the angles between the lines.

Q.E.D.

PROBLEMS SOLVED BY "INTERSECTION OF LOCI."

When the idea of a locus is clearly understood, two loci may often be employed in the solution of a problem. An example of this is seen in Ex. 1 on page 132. To take another example.

Ex. ABC is a triangle such that $AB = 6$ cm., $AC = 5$ cm., $BC = 7$ cm. Find all points that are equidistant from AB and BC, and at the same time are 4 cm. away from A.

If P is one of the required points—

- (i) P is a point equidistant from AB and BC; all such points must lie on the bisector of the angle B. Label this line "locus I." P lies somewhere on "locus I."

- (ii) P is also 4 cm. away from A; all such points must lie on a circle drawn with centre A and with a radius of 4 cm. Label this circle "locus II." P lies somewhere on "locus II."

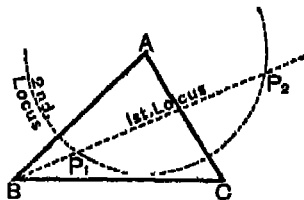


FIG. 109.

Hence P can only lie where "locus I." meets "locus II.," that is, at the two points marked P_1 and P_2 in the figure.

PRACTICAL EXERCISES ON LOCI.

EXAMPLES 51.

1. ABC is a triangle in which $BC = 2.5''$, $CA = 3''$, $AB = 1.8''$. Find points which are equidistant from A and C and which are $1''$ from B.

2. ABC is a triangle in which $AB = 8$ cm., $\angle A = 70^\circ$, $\angle B = 60^\circ$. Find points which are 5 cm. from A and 6 cm. from B.

3. Find the centre of a circle which has a radius of $1''$ and which passes through two points A and B $1.2''$ apart. Draw the circle.

4. In $\triangle ABC$ the angle A is a right angle, $AB = 2''$ and $AC = 1.8''$; D is the middle point of BC. Find a point P which is (1) as far from A as it is from B, and also (2) as far from C as it is from D. Measure the distance of P from A.

5. The $\triangle ABC$ is isosceles and in it $AB = AC = 2''$, $BC = 1.5''$. Find a point P which is (i) equidistant from the points A and B, and (ii) equidistant from the lines AC and BC. Measure PA.

6. A triangle has sides of 7 cm., 8 cm., 9 cm. Find a point inside it which is 2 cm. away from the shortest and longest sides of the triangle. How far away is it from the third side?

7. The sides of a triangle are 1.7", 1.8", 2.1". Find a point P which is equidistant from its vertices.

8. A tree is equidistant from two straight paths AB and AC, which meet at an angle of 30° ; the tree is 170 yards away from the straight path BC, which crosses the others so that $AB = 500$ yards and $AC = 250$ yards. Find all the possible positions of the tree by means of a figure drawn to scale, and find the possible distances of the tree from A.

9. Draw a triangle whose sides are 2", 2.1", 2.9". Find a point P inside it which is equidistant from the three sides.

THEORETICAL EXERCISES ON LOCI.

EXAMPLES 52.

1. A and B are fixed points, and APB is a triangle of constant area; what is the locus of P?

2. CD is a given straight line, and A and B are two points not in CD. Find a point P in CD such that $AP = BP$. When is this impossible?

3. A, B, C, D are four points such that $AC = AD$ and $BC = BD$. Prove that any point on the straight line AB is equidistant from C and D.

4. ABCD is an irregular quadrilateral. Find a point O such that $OA = OC$ and $OB = OD$.

5. ABC is a triangle. Find a point in CA equidistant from BA and BC.

6. AB and CD are two parallel lines met by a transversal PQ. Show how to find all the positions of a point which is equidistant from all three lines.

7. $\triangle ABC$ is of constant area on a fixed base BC; if O is the middle point of BC, find the locus of the middle point P of OA.

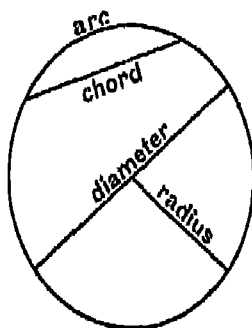
8. Find a point which is equidistant from the angular points of a given triangle.

9. AB and A'B' are two equal non-intersecting straight lines. Find a point O such that $OA = OA'$ and $OB = OB'$. Prove that $\angle AOB = \angle A'OB'$.

SECTION IV.

THE CIRCLE.

A circle is a plane figure bounded by a curved line, every point on which is at *the same distance* from a certain fixed point inside the figure. This point is called the *centre* of the circle, and every line from the centre to any point on the boundary is called a *radius*. The *curved boundary* of the circle is called the *circumference*. Any portion of the circumference is called an *arc*.

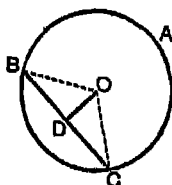


A chord of a circle is a straight line joining any two points on the circumference.

A diameter of a circle is a chord which passes through the centre

THEOREM 33.

The straight line drawn from the centre of a circle to bisect a chord is at right angles to the chord;
and, conversely, the straight line drawn from the centre of a circle at right angles to a chord bisects the chord.



Given a circle ABC with centre O, and a chord BC, and a line OD bisecting BC.

To prove that $OD \perp BC$.

Construction. Join OB, OC.

Proof. In the Δ s BOD, COD,

$$\left\{ \begin{array}{ll} OB = OC, & (\text{radii}) \\ BD = DC, & (\text{given}) \\ \text{and OD is common,} & \\ \therefore \Delta s \equiv, & (3 \text{ sides}) \\ \text{so that } \angle ODB = \angle ODC, & \\ \therefore \text{these are right } \angle s, & (= \text{adj. } \angle s) \\ \text{and } OD \perp BC. & \end{array} \right.$$

Q.E.D.

Conversely, given that $OD \perp BC$,

To prove that $BD = DC$.

Construction. Join OB, OC.

Proof. In the Δ s BOD, COD,

$$\left\{ \begin{array}{ll} \angle s BDO, CDO \text{ are right } \angle s, & (\text{given}) \\ OB = OC, & (\text{radii}) \\ \text{and OD is common,} & \\ \therefore \Delta s \equiv, & (rt. \angle, hyp., 1 \text{ side}) \\ \text{so that } BD = DC. & \end{array} \right.$$

Q.E.D.

Ex. Prove Part II of the above Theorem by Pythagoras.

Cor. 1. *The perpendicular bisector of a chord of a circle passes through the centre of the circle.*

This can be deduced as the result of Theorem 33, or it can be proved directly as follows:—

If BC is the chord of the circle (centre O), and DE the perp^r bisector of BC , then DE is the locus of all points equidistant from B and C ; but the centre O is equidistant from B and C ,
(equal radii)
 $\therefore O$ must lie on DE .

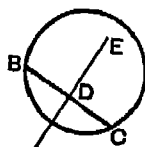


FIG. 176.

Cor. 2. *To find the centre of a given circle.*

Draw two chords AB , CD which are not $\parallel$; draw their perp^r bisectors, meeting in O . (See Fig. 177.)

Then, since the centre is on both perp^r bisectors, by Cor. 1 it follows that O must be the centre of the circle.

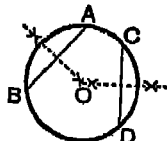


FIG. 177.

Cor. 3. *A circle is symmetrical about any diameter.*

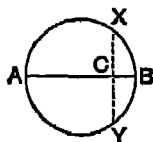


FIG. 178.

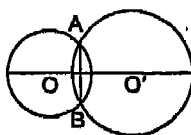


FIG. 179.

For if AB be a diameter and XY be any chord $\perp AB$ and meeting it at C , $XC = CY$,

$\therefore$ if the figure be folded over about AB , X will fall on Y ; so also for every pair of corresponding points on the circle.

Hence it follows that two circles intersecting at A and B are symmetrical about the line of centres OO' ;

$\therefore A$ will fold over to B about the line OO' ,

i.e. the common chord (AB) is bisected at right angles by the line of centres (OO'). (See also Ex. 12 on page 146.)

*†THEOREM 34.

There is one circle and only one circle which passes through three points not in the same straight line.

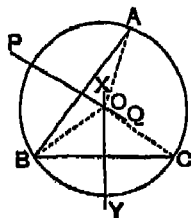
Given three points A, B, C not in the same straight line.

Required to prove that one, and only one, circle can be drawn through A, B, C.

Construction. Join AB, BC.

Draw PQ and XY, the perp^r bisectors of AB and BC; they must meet, since they are perp^r to lines which are inclined to one another; let them meet at O.

Draw a circle with centre O and radius OA.



This circle will also go through B and C.

Proof. Since O is on the perp^r bisector of AB, $OA = OB$; and since O is on the perp^r bisector of BC, $OB = OC$;

$$\therefore OA = OB = OC.$$

Hence the circle drawn with O as centre and radius OA will go through A, B and C.

And obviously no other circle is possible, for the perp^r bisectors of AB and BC meet in one point only, viz. O.
Q.E.F.

EXAMPLES 53.

NUMERICAL.

1. The radius of a circle is 26 ins.; find by calculation the length of a chord which is 10 ins. away from the centre.

2. Calculate the length of the perpendicular drawn from the centre of a circle, radius 2' 1'', on a chord 1' 2'' long.

3. AB and AC are two chords of a circle whose diameter is 5'', and their lengths are 2'' and 3''; calculate the distances of AB and AC from the centre of the circle. (Answer correct to .01'')

4. A suspension bridge in the form of an arc of a circle spans a river. The distance between the supports is 120 feet and the height of the top of the arch above the level of the tops of the supports is 20 feet. Calculate the radius of the circular arc.

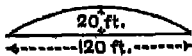


FIG. 180.

PROBLEM 7.

To circumscribe a circle to a given triangle.

Given a $\triangle ABC$.

Required to draw a circle through A, B, C.

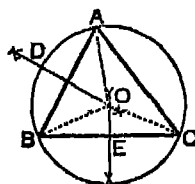


Fig. I

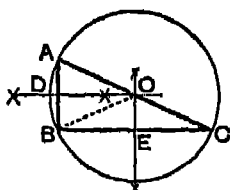


Fig. II

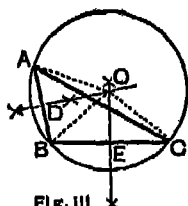


Fig. III

Construction. Draw the perp^r bisector DO of AB, cutting AB in D; draw the perp^r bisector EO of BC, cutting BC in E; let DO and EO meet in O.

Draw a circle with centre O and radius OB.

This circle will go through A, B, C.

Proof. Since O is on the perp^r bisector of AB,

$$OA = OB;$$

and since O is on the perp^r bisector of BC,

$$OB = OC;$$

$$\therefore OA = OB = OC;$$

hence the circle drawn with O as centre and radius OA will go through A, B, C.

Q.E.F.

EXAMPLES 53 (continued).

5. Two parallel chords are drawn in a circle of radius 17", and their lengths are 16" and 30". Find their distance apart if they are on (i) the same side of the centre; (ii) opposite sides.

6. Two chords of a circle are parallel and the centre of the circle is between them; they are 2' 11" apart and their lengths are 2' 6" and 3' 4". Calculate the radius of the circle and the distance of each chord from the centre. (Let the distance of one chord from the centre of the circle be x'' .)

(SHORTER.)

L

7. Two circles have the same centre and their radii are 3 in. and 5 in. A line $ABCD$ is drawn to meet the outer circle in A and D and the inner one in B and C . If BC is 1.8 ins., calculate the length of AD .

8. Find

- (i) the radius of a circle in which a chord of length $2a$ is at a distance b from the centre of the circle;
- (ii) the distance from the centre of a circle (radius r'') of a chord $2r'$ long.

THEORETICAL.

9. AB and AC are two equal chords of a circle drawn through a point A on the circumference. Prove that they make equal angles with the radius drawn to A .

10. Find the locus of the centres of circles which pass through two fixed points.

11. Two circles intersect in A and B and the centre C of AB is joined to P and Q , the centres of the circles. Show that PCQ is a straight line.

12. Prove that the common chord of two circles is bisected at right angles by the line of centres. [This can be proved by the Kite Proof; other proofs are suggested by the two previous Examples.]

13. Two concentric circles are drawn and a straight line $ABCD$ is drawn to cut one circle in A, D and the other in B, C . Prove that $AB = CD$. (What construction is necessary?)

14. Two circles meet in P and Q and a line $ABCD$ is drawn parallel to PQ to meet one circle in A, D , and the other in B, C . Prove that $AB = CD$. (See Fig. 181.)

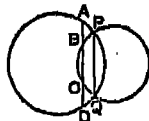


Fig. 181.

15. If two chords of a circle are parallel, prove that the line joining the middle point of one chord to the centre of the circle will bisect the other chord.

Deduce the locus of the middle points of a series of parallel chords of a circle.

16. Explain how to draw through a given point inside a circle a chord which shall be bisected at the given point. Add proof.

17. Two circles intersect in A and B , their centres being X and Y . Through A is drawn a line parallel to XY to meet one circle again in M and the other in N . Prove that $MN = 2XY$.

18. AB is a diameter of a circle and CD a chord that does not cut AB . If AE and BF are perpendiculars from A and B to CD (produced both ways), prove that $EC = DF$.

19. With a given point as centre, show how to draw a circle which shall cut a given circle at the ends of a diameter of the latter circle.

20. If two circles intersect at A and B and two parallel lines PAQ , XPY are drawn through A and B meeting one circle in P and X and the other in Q and Y , prove that $PQ = XY$.

21. Prove by means of Theorem 33 that the angle in a semicircle is a right angle. [Join the centre of the circle to the mid-point of one of the chords forming the right angle.]

PRACTICAL.

All the numerical examples (1-8) can be treated as practical questions and the figures drawn to scale, either to check the calculated answers or as separate questions.

22. Draw about two-thirds of a circle by tracing round the circumference of a penny. Find the centre of the arc and measure the radius in inches. Use compasses to complete the circle.

23. A skater cuts out an arc of a circle on the ice; the straight line joining the ends of the arc is 15 feet long and his furthest distance from this line is 6 feet. Find by drawing the radius of the circle.

24. Draw the circles which circumscribe the following triangles, and measure their radii:—

$$(i) a = 3'', b = 2.8'', c = 1.7'';$$

$$(ii) a = 1.8'', b = 2.4'', C = 105^\circ.$$

25. A , B and C are three points in order on a straight line; $AB = 5$ cm., $BC = 3$ cm. Construct a circle to pass through A and B and to have its centre 6 cm. from C . Measure the radius in mm.

26. Two circles have a common chord 1.4'' long and the distance between their centres is 2.5''. If the radius of one circle is 1.1'', draw the circles and measure the radius of the other circle.

27. A window consists of a rectangle surmounted by an arc less than a semicircle; its width is 2' 6'' and its total height up to the top of the arc is 4'. If the height of the rectangle is 3', find by drawing the radius of the arc.

THEOREM 85.

Equal chords of a circle are equidistant from the centre.

Given two equal chords AB , CD of a circle ABC whose centre is O .

To prove that the perpⁿ OX , OY from the centre O to the chords are equal.

Construction. Join OA , OC .

Proof.

$$\therefore OX \perp AB,$$

$$\therefore AX = \frac{1}{2} AB;$$

$$\text{Similarly } CY = \frac{1}{2} CD;$$

$$\text{but } AB = CD, \quad (\text{given})$$

$$\therefore \text{halving, } AX = CY.$$

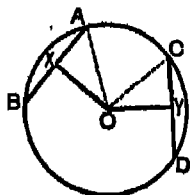
Then in the Δs AOX , COY ,

$$\begin{cases} \angle s \text{ } X \text{ and } Y \text{ are right angles,} & (\text{given}) \\ AO = CO, & (\text{radii}) \\ \text{and } AX = CY, & (\text{proved}) \end{cases}$$

$$\therefore \Delta s \equiv, \quad (\text{rt. } \angle, \text{ hyp., 1 side})$$

$$\text{so that } OX = OY.$$

Q.E.D.

**EQUAL AND UNEQUAL CHORDS.****EXAMPLES 54.**

1. Equal chords of a pair of equal circles are equidistant from the centres.

2. Calculate the distance from the centre of a circle to a chord of length $2a$ in a circle whose radius is r .

What is the locus of the centres of equal chords of a circle?

3. If two chords of a circle, crossing at a point X inside the circle, are equally inclined to the diameter through X , prove that the chords are equal.

4. If two equal chords of a circle meet when produced at O , prove that the distances from O of the corresponding ends of the chords are equal.

5. If AB and CD are two chords of a circle, centre O , and $AB > CD$, prove that $OX < OY$, where OX , OY are perpⁿ to AB , CD from O . (Use Pythagoras.)

6. Prove that, of two chords of a circle, the one which is nearer the centre is greater than one more remote.

THEOREM 36.

If two chords of a circle are equidistant from the centre, they are equal.

* Given two chords AB , CD of a circle ABC whose centre is O , such that the perp^{ns} OX , OY drawn to them from the centre O are equal.

To prove that $AB = CD$.

Construction.

Join OA , OC

Proof.

In the Δ s AOX , COY ,

$$\begin{cases} \angle s X \text{ and } Y \text{ are right } \angle s, & \text{(given)} \\ AO = CO, & \text{(radii)} \\ \text{and } OX = OY, & \text{(given)} \end{cases}$$

$$\therefore \Delta s \equiv, \quad (\text{rt. } \angle, \text{ hyp., 1 side})$$

so that $AX = CY$.

But $OX \perp AB$,

$\therefore AB = \text{twice } AX$.

Similarly $CD = \text{twice } CY$;

but $AX = CY$,

(proved).

$\therefore$ doubling, we get $AB = CD$.

Q.E.D.

Ex. Prove Theorems 35 and 36 by means of Pythagoras.

EXAMPLES 54 (continued).

7. Show that any chord of a circle which does not pass through the centre is less than the diameter.

8. Explain how to draw (i) the longest, (ii) the shortest chord of a circle through a given point inside the circle. Give reasons.

9. A point O is taken 3 cm. away from the centre of a circle (radius 4 cm.). Draw the longest and shortest chords possible through O . Measure their lengths and check by calculation.

10. AB and CD are two equal chords of a circle, and PQ are their middle points. Prove that PQ is equally inclined to AB , CD .

11. AB and PQ are equal chords in two circles (centres at C and O), the former circle having the longer radius; prove that AB is further from C than PQ is from O .

* Refer to Figure on page 148.

THEOREM 37.

The angle which an arc of a circle subtends at the centre is double of that which it subtends* at any point on the remaining part of the circumference.

Given a circle ABC, and an arc BC, $\angle BOC$ being the $\angle$ which the arc subtends at the centre O, and $\angle BAC$ the $\angle$ which it subtends at the point A on the remaining arc BAC.**

To prove that $\angle BOC$ is double $\angle BAC$.

Constr. Join AO, and produce it to D.

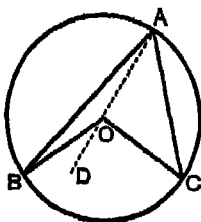


FIG. i.

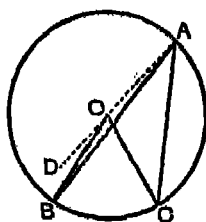


FIG. ii.

Proof. In $\triangle OAB$, $OA = OB$, (radii)
 $\therefore \angle OAB = \angle OBA$; (opp. equal sides)

but ext^r $\angle BOD$ of $\triangle OAB =$ sum of the int^r $\angle$ s OAB, OBA ,
 $\therefore \angle BOD$ is double $\angle OAB$.

Similarly $\angle COD$ is double $\angle OAC$.

Adding results in Fig. i and subtracting them in Fig. ii,
 $\angle BOC$ is double $\angle BAC$.

Q.E.D.

The above Theorem may be also enunciated thus:—

The angle at the centre of a circle is double the angle at the circumference standing on the same arc.

* See definition on page 151.

** In Fig. i the centre O is inside $\angle BAC$, and in Fig. ii the centre is outside $\angle BAC$.

N.B.—As the arc BC increases, the angle BOC which it subtends at the centre also increases. When the arc is half the circumference, the angle BOC is a straight angle (see Fig. iii); when the arc is larger still, the angle BOC is reflex (see Fig. iv). But the proof for Fig. i still holds for Figs. iii and iv, so that the truth in Theorem 37 always holds good.

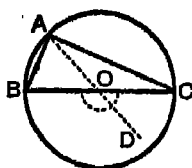


Fig. iii.

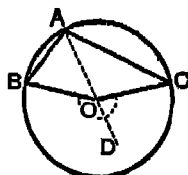


Fig. iv.

The angle subtended at a point C by two points A and B (or by the line AB) is the angle ACB. (See Fig. 182.) Suppose the eye is at C and looks at A and B; $\angle ACB$ is at the eye C.



Fig. 182.

Ex. In the figures on page 150, state the angles subtended (i) at A by BO, OC, arc BC, and (ii) at O by AB, AC, arc BAC.

A segment of a circle is the area bounded by a chord of the circle and an arc which the chord cuts off.

An angle in a segment is the angle subtended by the chord of the segment at any point on the arc.

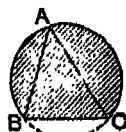
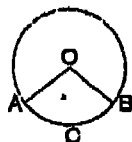


FIG. 183A.
Segment and angle in a segment.

A chord cuts off two segments from a circle, and the larger of these is called a major segment, and the smaller a minor segment. So too their arcs are called major and minor arcs. (Compare the segments above and below BC in Fig. 183A.)



Sector.
Fig. 183B.

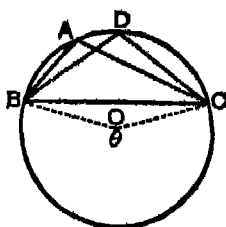
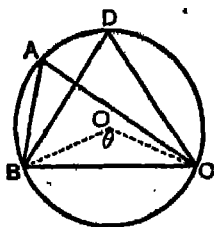
A sector of a circle is the area bounded by two radii of the circle and an arc which the radii cut off. (Fig. 183B.)

THEOREM 38.

Angles in the same segment of a circle are equal.

Given a segment of BAC of a circle, in which BAC and BDC are two angles.

To prove that $\angle BAC = \angle BDC$.



Constr. Join B and C to the centre O .

Proof. The arc BC subtends $\angle \theta$ at the centre O and $\angle BAC$ at the point A on the circumference.

$$\therefore \angle \theta = \text{double } \angle BAC;$$

Similarly $\angle \theta = \text{double } \angle BDC$.

Hence $\angle BAC = \angle BDC$. (*halves of the same $\angle, \theta$*)

Q.E.D.

Cyclic figures and Concyclic points. If a rectilineal figure is such that a circle can pass through its vertices, the figure is said to be *cyclic*; the angular points are, however, said to be *concyclic*. Thus in the figure to Theorem 40, the quadrilateral $ABCD$ is said to be *cyclic*; the four points A, B, C, D are, however, said to be *concyclic*.

EASY EXERCISES ON THEOREMS 37, 38.

EXAMPLES 55.

1. Which of the angles p, q, x, y in Fig. 184 are equal? Give reasons for your statements.



Fig. 184.



Fig. 185:

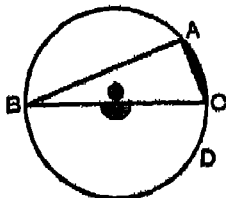
2. In Fig. 185 find the size of the angles a, b, c .
3. ABC is a triangle inscribed in a circle, centre O . Join OA, OB, OC . If $\angle BOC = 150^\circ$, $\angle COA = 136^\circ$, calculate the three angles of $\triangle ABC$.
4. If in Fig. 1 to Theorem 38, $\angle OBC = 36^\circ$, calculate $\angle s$ BOC, BAC, BDC .
5. What angle in Theorem 38, Fig. 1, is equal to $\angle ABD$? If AD is joined, what angle equals $\angle DAC$?
6. The diagonals of a cyclic quadrilateral $ABCD$ meet in O ; if $\angle BAO = 62^\circ$, $\angle BCO = 34^\circ$, $\angle BOC = 82^\circ$, find $\angle CAD$ and $\angle OCD$.
7. $ABDC$ is a quad^l and $BA = BC = BD$; if $\angle ACD = 132^\circ$, find $\angle ABD$.
8. In Fig. 1 of Theorem 38 prove that $\angle A$ is the complement of $\angle OBC$.
9. $ABCD$ is a cyclic quadrilateral, AC and BD meet in O , and AB and DC meet, when produced, in X . Prove that—
(i) $\triangle s$ AOD, BOC are equiangular, and (ii) so are $\triangle s$ XAO, XBD .
10. If O is the centre of the circle circumscribing a $\triangle ABC$, and OB, OC are found to bisect the angles B and C , prove that $\angle A$ must $= 60^\circ$.
- ✓11. ABC is a triangle, D is any point in BC , and X, Y are the centres of the circumcircles of the $\triangle s$ ABD, ACD . If $\angle AXD = 78^\circ$ and $\angle AYD = 96^\circ$, find $\angle BAC$.

THEOREM 89.

The angle in a semicircle is a right angle.

Given a circle ABD, centre O, and an angle BAC in the semicircle BAC.

To prove that $\angle BAC$ is a right $\angle$.



Proof. The arc BDC subtends the straight $\angle BOC$ at the centre O and the $\angle BAC$ at the point A of the arc BAC,

$$\begin{aligned}\therefore \angle BAC &= \text{half the str. } \angle BOC \\ &= \text{a right } \angle.\end{aligned}$$

Q.E.D.

(See also a proof by means of Book I, in Theorem 11, Cor. 2, on page 74.)

Cor. *The angle in a segment greater than a semicircle is less than a right angle, and the angle in a segment less than a semicircle is greater than a right angle.*

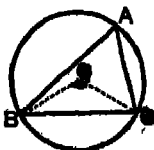


FIG. 186.

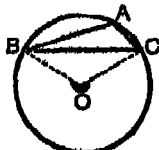


FIG. 187.

[For in Fig. 186 $\angle A$ is half the $\angle BOC$ at the centre;
but $\angle BOC$ is < 2 right $\angle$ s,
 $\therefore \angle A < 1$ right $\angle$.

Also in Fig. 187 $\angle A$ is half the reflex $\angle BOC$ at the centre
but reflex $\angle BOC > 2$ right $\angle$ s,
 $\therefore \angle A > 1$ right $\angle$.]

12. If in Fig. 189 $b + c = k + h$, prove that $x = 180^\circ - 2b$.

ANGLES IN A CIRCLE.

EXAMPLES 57.

NUMERICAL.

1. A regular pentagon is inscribed in a circle; how many degrees are there in an angle in any one of the five segments outside the figure?

2. BA, CD are chords of a circle meeting when produced at O; if $\angle AOD = 30^\circ$ and $\angle ACD = 42^\circ$, calculate $\angle CDB$.

3. AB is a diameter of a circle, and CD a chord $\perp$ AB; if $\angle ACD = 27^\circ$, calculate the angles ABD, CDB, ADC.

4. ABC is a triangle described in a circle, and $\angle B = 70^\circ$ and $\angle C = 40^\circ$. If X is a point on the circle on the opposite side of BC to A such that $XB = XC$, find the angles of $\triangle XBC$.

5. ABO is a straight line and $AB = 2''$, $BO = 1''$; also BAD ($= 40^\circ$) and ABC ($= 60^\circ$) are angles on the same side of ABO. Explain how to find points D, C on AD, BC respectively, making a cyclic quadrilateral ABCD such that DC produced goes through O.

6. In the figure to Ex. 11 on page 153 $\angle XDY = 98^\circ$; calculate $\angle BAC$.

THEORETICAL.

7. ABC is a $\triangle$, obtuse angled at B, having $AB = BC$. A circle, passing through A and C, cuts AC at D and CB produced at E. Prove that $DC = DE$.

8. Two circles meet in P, Q. Two lines APB, CQD are drawn to meet one circle in A, C, and the other in B, D. Prove that $\triangle s$ PCD, QAB are equiangular.

9. Show how to divide a circle into two segments so that the angle in one segment is four times the angle in the other segment.

10. AB and CD are parallel chords of a circle, and BC, AD meet at O (inside the circle). Prove that $OC = OD$.

11. A and D are points on a circle on the same side of any chord BC; prove that $\angle ABC + \angle ACB = \angle DCB + \angle DCB$.

12. AB and CD are chords of a circle cutting at O, and the angle which AB subtends at C is equal to the angle which DC subtends at B. Prove that $OC = OB$.

13. ABCD is a quadrilateral in which $AB = AC = AD$; prove that $\angle BAD + 2 \angle BCD = 360^\circ$.

14. ABC is a Δ and two circles are drawn through B and C, one of them cutting AB, AC at D, E and the other cutting AB, AC at F, G. Prove that $DE \parallel FG$.

15. APB, AQB are segments of circles on the same side of the chord AB and P is any point on the outer arc APB. If AP cuts the inner arc in Q, prove that $\angle PBQ$ is constant in size.

16. ABCD is a cyclic quadrilateral, the centre O of the circle being inside the figure. If the diagonals meet in X, prove that $\angle AQB + \angle COD = 2 \angle AXB$.

17. If one circle be drawn on a radius of another circle as diameter, any chord of the larger circle drawn through the point common to the circles is bisected by the smaller circle.

18. Two circles intersect in A and B, and XAY, PAQ are two lines meeting one circle in X, P and the other in Y, Q. Prove that XP and YQ subtend equal angles at B.

19. Two circles intersect at A and B; BDE is a chord of one circle, meeting the other in D, and AFG a chord of the second circle, meeting the first in F. Prove that EF and DG are parallel.

20. If a hexagon ABCDEF be inscribed in a circle, prove that $\angle A + \angle C + \angle E = 360^\circ$.

21. If one diagonal AC of a cyclic quadrilateral bisects the angles A and C, prove that angles B and D are right angles.

22. ABCD is a cyclic quadrilateral, in which $AB = \text{diag. AC}$; if AD, BC when produced meet in O, show that $\angle ACD = \angle O$.

23. BC is an arc of a circle, centre O, and BAC an angle at the circumference, with O inside it. If BO produced meets AC in D, and CO produced meets AB in E, prove that $\angle BDC + \angle BEC = \text{three times } \angle A$.

24. The internal bisectors of the angles B, C of ΔABC meet the circle ABC in Q, R. If $BR \parallel CQ$, prove that $\angle BAC = 60^\circ$.

25. If on the same side of a line AB as base two arcs are drawn, containing angles of x° and y° where $x > y$, show that AB subtends at any point P in the area between the arcs an angle which lies between x° and y° .

26. The chords CB, ED of a circle, centre O, are produced to meet in X; show that $\angle COE - \angle X = \angle CYE$, where Y is the point at which BE and CD meet.

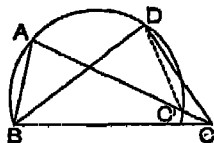
THEOREM 41.

If the straight line joining two points subtends equal angles at two other points on the same side of the line, the four points are concyclic.

Given a straight line BC which subtends equal angles at the two points A and D on the same side of BC .

To prove that A, B, C, D are concyclic.

Proof. Suppose that a circle is drawn through BAD , and that it misses C .



Then AC is a line inside the circle from the point A on the circle, and it will therefore meet the circle again, either before or after C .

Let it meet the circle again in C' (before C). Join DC' .

Then $\angle A = \angle BDC'$, ($\angle$ s in same segment)
but $\angle A = \angle BDC$; (given)

$\therefore \angle BDC' = \angle BDC$, which is absurd.

$\therefore$ the circle BAD cannot miss C ,
that is, A, B, C, D are concyclic.

Q.E.D.

Alternative Proof. Suppose that a circle is drawn through A, B, C , and that it misses D .

Take a point X on BC , and join XD ; then since X is inside the circle, XD , or XD produced, will cut the circle again, say at D' ; join $D'B, D'C$.

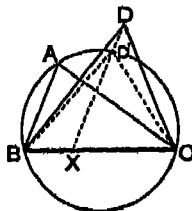
Then $\angle BAC = \angle BD'C$; (in same segment)
but $\angle BAC = \angle BDC$, (given)

$\therefore \angle BD'C = \angle BDC$,

which is absurd, since the sum of the base $\angle$ s of $\triangle D'BC$ is not equal to the sum of the base $\angle$ s of $\triangle DBC$.

$\therefore$ the circle ABC does not miss D ,
that is, A, B, C, D are concyclic.

Q.E.D.

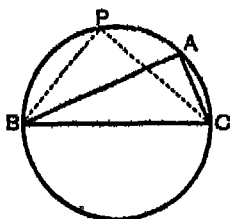


THEOREM 42.

The circle described on the hypotenuse of a right-angled triangle as diameter passes through the opposite vertex.

Given a $\triangle ABC$ right-angled at A , and a circle drawn on the hypotenuse BC as diameter.

To prove that the circle goes through A .



Constr. Take any point P on the circle; join PB , PC .

Proof. $\angle P$ is a right $\angle$. ($\angle$ in a semicircle)
and $\angle A$ is a right $\angle$, (given)
 $\therefore \angle P = \angle A$;

that is, BC subtends equal $\angle$ s at P and A ;

$\therefore$ a circle will pass through P , A , B , C ,

$\therefore A$ lies on the circle PBC , i.e. the circle on BC as diameter.
Q.E.D.

Cor. 1. The middle point of the hypotenuse of a right-angled triangle is equidistant from the right angle and from the extremities of the hypotenuse.

Thus if O is the middle point of the hypotenuse BC , $OA = OB = OC$; for O is the centre of the circle described on BC as diameter, and this circle goes through A .



FIG. 180.

Cor. 2. The locus of the vertices of all right-angled triangles drawn on a given line as hypotenuse is the circle described on the given line as diameter.

For this circle goes through A_1 , A_2 , A_3 ,
 $\therefore$ by Theorem 42.

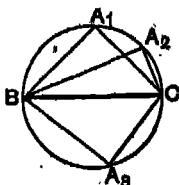


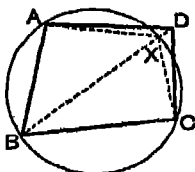
FIG. 191.

THEOREM 48.

If a pair of opposite angles of a quadrilateral are supplementary, its vertices are concyclic.

Given a quad^l ABCD in which the opp. $\angle$ s A and C are supplementary.

To prove that A, B, C, D are concyclic.



Proof. Let a circle be drawn through A, B, C; join BD. Then, if the circle does not pass through D, the line BD, which lies inside the Δ ABC, or the line BD produced, must cut the circumference.

Let this point of intersection be X; join XA, XC.

Then, since ABX is a cyclic quad^l,

$$\angle XAB + \angle XCB = \text{two right } \angle\text{s};$$

$$\text{but } \angle DAB + \angle DCB = \text{two right } \angle\text{s}, \quad (\text{given})$$

$$\therefore \angle XAB + \angle XCB = \angle DAB + \angle DCB,$$

i.e. the part is equal to the whole, which is absurd.

$\therefore$ the circle ABC must pass through D,

i.e. A, B, C, D are concyclic.

Q.E.D.

Alternative Proof. Let a circle be drawn through B, A, D.

Let P be any point on the circumference on the same side of the chord BD as C; join PB, PD.

Then, since DABP is a cyclic quad^l,

$\angle$ s P and A are supplementary,

but $\angle$ s C and A are supplementary,

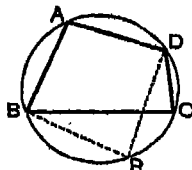
(given)

$$\therefore \angle P = \angle C,$$

$\therefore$ B, P, C, D are concyclic.

i.e. C lies on the circle BPD, i.e. on the circle ABD;

$\therefore$ A, B, C, D are concyclic.



(SHORTER.)

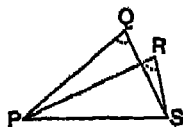
Q.E.D.

M

CONCYCLIC POINTS.

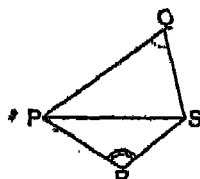
From Theorems 38, 39, 42, and 43 we conclude that four points P, Q, R, S lie on a circle:—

- (i) When the line joining any two (say P, S) subtends equal angles at the other two (Q, R), provided the latter two are on the same side of the line PS . (See Fig. 192.)
 or (ii) When the line joining any two (say P, S) subtends supplementary angles at the other two (Q, R), provided the latter two are on opposite sides of the line PS . (See Fig. 193.)



$$\angle Q = \angle R.$$

FIG. 192.



$$\angle Q + \angle R = 180^\circ.$$

FIG. 193.

If the angles Q and R are both 90° , it does not matter on which side of PS they lie, for by Theorem 42, the circle on PS as diameter will go through both, and the centre of the circle will be at O , the middle point of PS . (See Fig. 191.)

The above tests (i) and (ii) are closely related to each other, for if PS be any chord of a circle and Q be a point moving round the

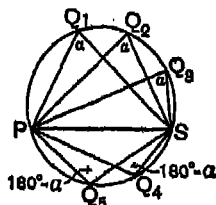


FIG. 194.

circle, then the angle which PS subtends at Q is constant (say α) as long as Q is on one side of PS and changes to $180^\circ - \alpha$ and remains constant at that value when Q passes to the other side of PS . Note again that the value of the angle does not change as Q changes sides when $\alpha = 90^\circ$.

CONCYCLIC POINTS.

EXAMPLES 58.

1. ABC is a triangle, and BD , CE are drawn perpendicular to AC , AB , meeting in O . Give reasons why—

- (i) the four points B , E , D , C are concyclic;
- (ii) the four points A , E , O , D are concyclic;
- (iii) $\angle EBD = \angle ECD$.

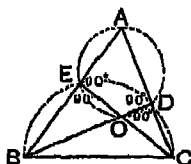


Fig. 195.

2. Discuss which of the following figures are cyclic:—

- (i) square, (ii) parallelogram, (iii) rectangle,
- (iv) rhombus, (v) kite.

3. ABC is a triangle and PQ is drawn to cut AB in P and AC in Q so that $\angle APQ = \angle C$; prove that P , Q , C , B are concyclic.

4. Show that a line parallel to the base of an isosceles triangle cuts off from the triangle a quadrilateral which is cyclic.

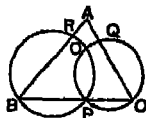


Fig. 196.

5. The circle BPO in Fig. 196 cuts AB in R , and the circle CPO cuts AC in Q ; show that A , Q , O , R are concyclic.

6. A semicircular protractor slides in its own plane with the ends of its base always on two fixed perpendicular lines OA , OB . Prove that any mark M on the rim (say the 20° mark) will move on a straight line through O . What are the angles between the locus of M and the fixed lines?

7. C and D are the points of intersection of two circles whose centres are A and B ; BC is joined and produced to meet the first circle in E . Prove that A , E , B , D are concyclic.

8. Two circles cut at A and B ; two lines PAQ , XAY are drawn through A to meet one circle in P , X and the other in Q , Y . If XP and YQ are produced to meet in O , show that O , X , B , Y are concyclic.

9. $ABCD$ is a cyclic quadrilateral and AB is produced to X ; if O is the point of intersection of the bisectors of the angles XBC and CDA , prove that O lies on the circle $ABCD$.

EQUAL ANGLES AND EQUAL ARCS.

If in a circle, centre O , equal angles AOB and COD are taken at the centre, it is evident from the uniform shape of a circle that the arc $AB =$ the arc CD (Fig. 197).

So also in two equal circles (Fig. 198).

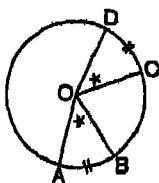


Fig. 197.

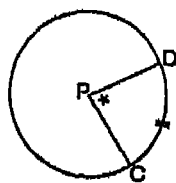
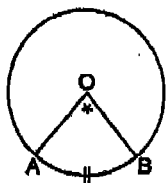
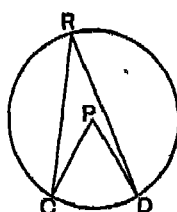
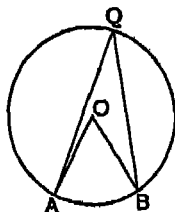


Fig. 198.

Thus we see the truth of the theorem that follows.

***|THEOREM 44.**

In equal circles, or in the same circle, equal angles at the centre stand on equal arcs, and, conversely, equal arcs subtend equal angles at the centre.



Thus in the figure, where the circles are equal,

if $\angle AOB = \angle CPD$, arc $AB =$ arc CD ,

and if arc $AB =$ arc CD , $\angle AOB = \angle CPD$.

Also if angles AQB , CRD are taken on the circumference,

$\angle AOB$ is twice $\angle AQB$, $\angle CPD$ is twice $\angle CRD$,

$\therefore$ if arc $AB =$ arc CD , $\angle AQB = \angle CRD$.

Hence the Corollary:—

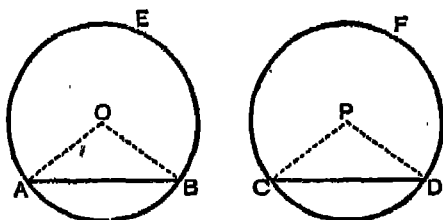
In equal circles, or in the same circle, equal angles at the circumference stand on equal arcs, and, conversely, equal arcs subtend equal angles at the circumference.

***THEOREM 45.**

In equal circles, or in the same circle, equal chords cut off equal arcs. Conversely in equal circles, or in the same circle, the chords of equal arcs are equal.

Given two equal circles ABE, CDF, centres O and P, and two equal chords AB, CD.

To prove that arc AB = arc CD, and the remaining arc AEB = the remaining arc CFD.



Constr. Join OA, OB; PC, PD.

Proof. In Δ s OAB, PCD,

$$\begin{cases} OA = PC, & \text{(equal radii)} \\ OB = PD, & \text{(equal radii)} \\ AB = CD, & \text{(given)} \end{cases}$$

$\therefore \Delta s \equiv$, (three sides)

so that $\angle O = \angle P$,

$\therefore$ arc AB = arc CD. (subtended by $\angle$ s)

Conversely. If arc AB = arc CD, to prove that chord AB = chord CD.

Proof (in brief). Arc AB = arc CD,

$\therefore \angle O = \angle P$, (standing on = arcs)

$\therefore \Delta s$ OAB, PCD $\equiv$, (2 sides and incl. $\angle$)

so that chord AB = chord CD.

Q.E.D.

N.B. In Theorem 45, Part I, it is necessary to take both arcs minor (i.e. less than a semi-circumference) or both arcs major (i.e. greater than a semi-circumference).

Thus in Fig. 199 if chord $AB =$ chord CD , it follows that arc $AXB =$ arc CQD (both minor arcs) and arc $AYB =$ arc CPD (both major arcs).

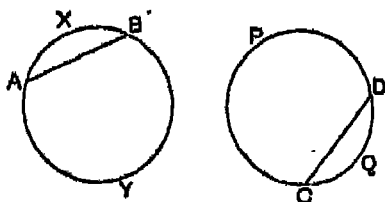


Fig. 199.

EQUAL ARCS.

EXAMPLES 59.

1. Find the angle subtended at the centre of a circle by a chord PQ equal to the radius; what fraction is the arc PQ of the whole circumference?

2. Two equal circles, PAR , QAS meet in A and B , and two lines PAQ , RAS are drawn through A ; prove that arc $PR =$ arc QS and chord $PR =$ chord QS .

3. If B is the middle point of any arc ABC of a circle, prove that chord $AB =$ chord BC .

4. If AB is a chord of a circle cutting off two segments ACB , ADB , prove that the bisector of any angle in the segment ACB goes through the middle point of the arc ADB .

5. The bisectors of all the angles in a given segment of a circle all go through one point. Show that the point of concurrency is equidistant from the ends of the base of the segment.

6. AB and CD are two parallel chords of a circle, A and C being corresponding extremities; show that (i) arc $AC =$ arc BD ; (ii) chord $AC =$ chord BD .

[The second of the above results is often met with; besides the proof by means of equal angles, there are other proofs, e.g. :—

(a) join AD and BC and use congruent Δ s;

(b) produce AC and BD to meet and use isosceles Δ s.]

7. In the figure to Ex. 6 prove that arc $ABD =$ arc BAC and chord $AD =$ chord BC ,

8. PQ and PR are a chord and a diameter through a point P on a circle; prove that the radius $\parallel PQ$ will bisect the arc QR .

9. If two chords of a circle (or of equal circles) subtend supplementary $\angle$ s at any point on the circumference, they are equal.

10. Through the points of intersection of two circles, two parallel lines are drawn as far as the circumferences; prove that

the straight lines joining corresponding extremities are equal and parallel.

11. Two equal circles cut in A, B ; PAQ is any straight line through A meeting one circle in P and the other in Q ; prove that $BP = BQ$.

12. If two arcs of a circle subtend complementary angles at any point of the circumference, their sum is a semi-circumference.

13. If AB, BC, CD are three consecutive sides of an equilateral polygon inscribed in a circle, prove that $\angle B = \angle C$.

14. If $\angle AOB, \angle COD$ are angles at the centre O of a circle $ABCD$, such that $\angle AOB = \text{twice } \angle COD$, prove that (i) arc $AB = \text{twice arc } CD$, (ii) chord AB is not twice chord CD .

15. X and Y are the middle points of the unequal arcs AB, AC of a circle; if XY cuts AB in D and AC in E , prove that $AD = AE$.

16. $ABCDE$ is a pentagon inscribed in a circle; $AB = AE$ and $\angle A = 98^\circ$, also $BC = CD = DE$. Calculate the $\angle$ s of the figure.

17. ABC is a Δ , A being an acute angle; CZ, BX are drawn at right angles to AB, AC and are produced to meet the circle ABC in K, Y . Prove that $AY = AK$.

18. Straight lines are drawn on the dial of a watch joining the positions I and V, II and X, III and VIII, VII and XI; prove that the figure enclosed by these lines is a square.

19. If equal angles at the centres of two circles stand on equal chords, prove that the circles are equal.

20. In the ΔABC , $AB = AC$; if D is a point on BC produced, prove that the circles ABD, ACD are equal. (N.B. Circles are equal if equal chords subtend equal angles, either at the centres, or at the circumferences.)

21. In the ΔABC , $AB = AC$; if D is any point in BC , prove that the radii of the circles ABD, ACD are equal.

22. ABC is an equilateral triangle inscribed in a circle; prove that the line joining the middle points of the minor arcs AB, AC is trisected by AB and AC .

23. AC and BD are two $\perp$ chords of a circle; prove that arc $AB + \text{arc } CD = \text{arc } AD + \text{arc } BC = \text{half the circumference}$.

24. If two chords (AB, CD) intersect (at O) inside a circle, they form an angle ($\angle AOC$) equal to that subtended at the centre by half the sum of the arcs (AC, BD) they cut off.

25. AC and BD are two fixed diameters of a circle, centre O ; P is any point on the circumference, and PM, PN are the perpendiculars drawn from P to AC, BD . Prove that the line MN is of constant length for all positions of P .

TANGENT TO A CIRCLE.

(WITHOUT THE USE OF LIMITS.)

A secant to a circle is a straight line cutting the circle in two points.

A tangent to a circle is a straight line which meets the circle in one point only, however far it is produced; the point of meeting is called the point of contact and the tangent is said to touch the circle. ✓

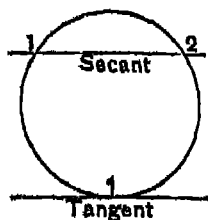


FIG. 200.

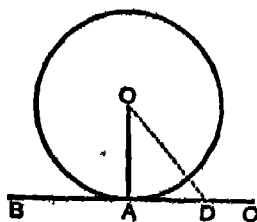
Ex. By drawing a series of parallel secants all perpendicular to the same diameter AB , it can be seen that at each end of the diameter is a line meeting the circle in one point only (A or B). Such a line is a tangent. What do you think is its relation to the diameter AB ?

THEOREM 46.

The straight line drawn perpendicular to a radius of a circle at its extremity is a tangent to the circle.

Given a circle, centre O , and a radius OA , and a line BAC perp^r to OA at A .

To prove that BAC is a tangent to the circle at A .



Constr. Take D , any other point in BC besides A ; join OD .

Proof. In $\triangle OAD$, $\angle A = 90^\circ$, (given)
 $\therefore \angle ODA < 90^\circ$, ($\angle s$ of a $\triangle$)
 $\therefore \angle A > \angle ODA$,
 $\therefore OD > OA$, a radius,
 $\therefore D$ must be outside the circle.

So also every other point in BC besides A is outside the circle.

$\therefore BC$ meets the circle in one point only.
i.e. BC is a tangent to the circle at A .

Q.E.D.

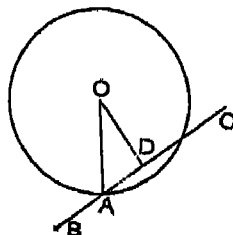
Cor. 1. A tangent to a circle is perpendicular to the radius drawn to the point of contact.

[Thus if BAC is a tangent at A to the circle, centre O , BAC will be perp^r to OA .

For if not, draw $OD \text{ perp}^r$ to BAC .
 Then $\angle D$ is a right $\angle \therefore \angle A < \angle D$,
 $\therefore OD < OA$, a radius,
 $\therefore D$ must be inside the circle,
 $\therefore AD$ produced, *i.e.* BAC , will meet the circle again.

But this is absurd, since BAC is a tangent.

$\therefore BAC$ is perp^r to OA .]



Cor. 2. At every point on a circle one and only one tangent can be drawn.

To draw a tangent to a circle at A , draw $BAC \text{ perp}^r$ to radius OA .

Cor. 3. The perpendicular to a tangent at its point of contact passes through the centre.

For the radius drawn to the point of contact is this perpendicular.

TANGENT TO A CIRCLE

(ALTERNATIVE METHOD, USING LIMITS).

Take a secant AB cutting a circle in P and Q . Regard P as a fixed point and the secant as revolving about P (just as if AB were

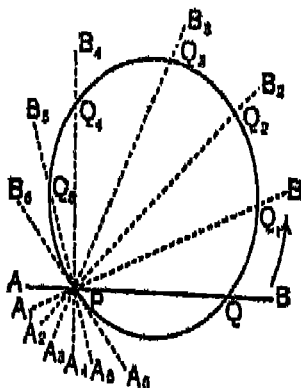


FIG. 201.

fastened to a wire by a universal hinge at P). As AB is moved round, the point Q approaches P , and ultimately it reaches P . When the two points P and Q coincide, the line AB is said to be a tangent to the circle, and P is called its point of contact. (In Fig. 201 the tangent at P is $A_6 B_6$.)

A tangent to a circle is the line which a secant becomes when the two points in which it meets the circle are allowed to coincide.

THEOREM 46.

(ALTERNATIVE PROOF.)

A tangent to a circle is perpendicular to the radius drawn to the point of contact.

Given a circle CDE, centre O, and a tangent to the circle.

To prove that the tangent is perp^r to the radius to the point of contact.

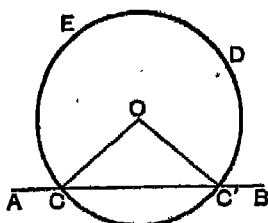


Fig. i.

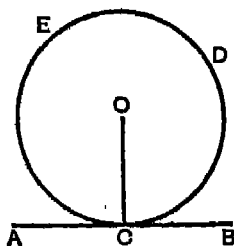


Fig. ii.

Constr. Take a secant AB cutting the circle in C and C'; join OC and OC' (Fig. i).

Proof. In $\triangle OCC'$, $OC = OC'$, (radii)

$$\therefore \angle OCC' = \angle OC'C,$$

$\therefore$ their supplements are equal,

$$\text{i.e. } \angle OCA = \angle OC'B,$$

Now move AB until C' coincides with C;

then AB will be a tangent (Fig. ii),

$$\text{and } \angle OCA = \angle OCB;$$

$\therefore$ each of these $\angle$ s is a right $\angle$, (= adj. $\angle$ s.)

and the tangent AB is perp^r to rad. OC.

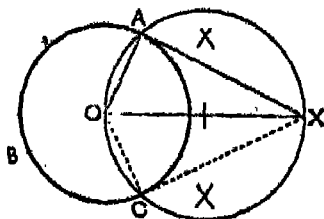
Q.E.D.

Cor. 1. The straight line drawn perpendicular to a radius of a circle at its extremity is a tangent to the circle.

Cor. 2. } As on page 169.
Cor. 3. }

PROBLEM 8.

To draw a tangent to a circle from a given external point.
 Given a circle ABC , centre O , and a point X outside it.
 To draw from X a tangent to the circle ABC .



Constr. Join OX and bisect it; on OX as diameter draw a circle OAX cutting the given circle in A and C .
 Join XA .

Then XA shall be a tangent to the circle ABC .

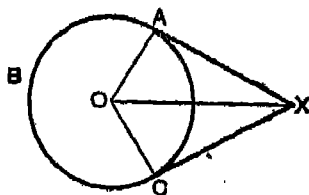
Proof. If OA is joined, $\angle A$ is a rt $\angle$; (*in a semicircle.*)
 $\therefore XA$ is a tangent to the circle. (*igl. $\perp$ rad.*)
 Q.E.F.

N.B. A second tangent, XC , can also be drawn from X to the circle ABC , and no other tangent is possible. Hence—

Cor. Two, and only two, tangents can be drawn to a circle from a given external point.

THEOREM 47.

The two tangents drawn to a circle from an external point are equal.



Given a circle ABC , centre O , and two tangents XA , XC drawn to it from the point X .

To prove that $XA = XC$.

Constr. Join OA , OC , OX .

Proof. In $\triangle XAO$, XCO ,

$$\begin{cases} \angle s \text{ A and C are right } \angle s, & (\text{igt. } \perp \text{ rad.}) \\ OX \text{ is common,} \\ OA = OC, & (\text{radii}) \end{cases}$$

 $\therefore \triangle s \equiv$, (rt. $\angle$, hyp. 1 side)
 so that $XA = XC$.

Q.E.D.

Cor. *The two tangents drawn to a circle from an external point*

- (i) *subtend equal angles at the centre,*
(i.e. $\angle AOX = \angle COX$)
- (ii) *are equally inclined to the diameter through the point.*
(i.e. $\angle AXO = \angle CXO$)

Ex. Prove Theorem 47 by means of Pythagoras.

TANGENTS.

EXAMPLES 60.

NUMERICAL.

- Find the length of a tangent drawn to a circle of radius 7" from a point 25" away from the centre of the circle.
- The tangents to a circle from a point X are 2 ft. long and the radius of the circle is 1' 6". Calculate the distance of X from the centre of the circle.
- Two concentric circles are drawn whose radii are 2" and 1' 6". Calculate the length of any chord of the outer which touches the inner.
- Inside a circle, radius a , are drawn a series of equal chords each of length $2b$. Calculate the distance of each chord from the centre of the circle. What is the locus of the middle points of the chords? Find the connection between this locus and each chord.

5. A number of tangents are drawn to a circle (centre O and radius $2''$), and a point P is taken on each at a distance of $1''$ from the point of contact. Calculate the length of OP . What is the locus of P ?

6. A road is bounded by two walls AEB , CFD in the shape of two concentric quadrants of circles (centre of each O), the radius of the smaller being 60 yds. If it is just possible to see D from C , find BD , the width of the road.

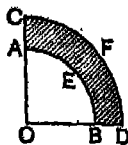


Fig. 202.

PRACTICAL.

(Ex. 1, 2, 3, 5, 6 above can also be solved graphically.)

7. Draw a series of equal chords in a circle; what seems to be the locus of their middle points?

8. Draw two tangents to a circle (radius $1''$), meeting at an angle of 70° . Measure their lengths.

9. Draw two tangents to a circle (radius $1\frac{1}{2}''$) which meet at an angle of 60° . Measure them. (Do not use protractor.)

10. AB is a line 6 cm. long. Draw a circle with A as centre so that the length of the tangent drawn to it from B is $3\frac{1}{2}$ cm. Measure the radius.

THEORETICAL.

11. The tangents at the two extremities of a diameter of a circle are parallel.

12. If two parallel tangents are drawn to a circle, their points of contact are in a straight line with the centre of the circle.

13. If two tangents AB , AC are drawn to a circle, centre O , show that the chord of contact BC is bisected at right angles by AO .

14. Prove that all chords of a circle which touch a smaller concentric circle are equal.

15. Find the locus of the middle points of a series of equal chords of a circle. Show that the chords are tangents to the locus.

16. If AB and CD , two parallel tangents to a circle whose centre is O , are met by a third tangent BED at B , D , prove that ED subtends a right angle at O .

17. A diameter AB of a circle bisects all chords which are parallel to the tangent at A .

18. ABCD is a quadl which circumscribes a circle whose centre is O. Prove that

$$AB + CD = AD + BC.$$

19. State and prove the converse to Ex. 18.

20. In Fig. 203, prove that $\angle AOB + \angle COD = \text{two right angles}$.

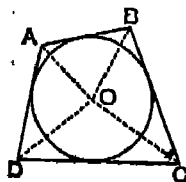


FIG. 203.

21. If a parm circumscribe a circle, it will be a rhombus.

22. If AB is a chord of a circle and AC is a diameter, and if the tangents at A and B meet at D, prove that $\angle ADB = 2 \angle BAC$.

23. Find the locus of points from which the tangents to a given circle have a given length.

24. AX and AY are tangents to a circle, and a third tangent, at D, meets AX and AY, not produced, in B and C. Prove that the perimeters of Δs ABD, ACD are equal.

25. How can a chord be drawn in a given circle, through a given point P, which shall be equal to a given length AB? (Draw any chord equal to AB and use Ex. 15.)

26. Find a point in a given line (XY) from which the tangent to a given circle is equal to a given length (PQ). (Use Ex. 23.)

PRACTICAL.

(Examples 14, 15, and 23 are sometimes useful in connection with practical problems: compare Examples 25 and 26.)

27. A is a point 1.2" from the centre of a given circle (radius 1.8"). Draw a chord through A which is 3" long. Measure the parts into which the chord is divided at A.

28. A circle is drawn with a radius of 5 cm. and any line XY is drawn outside the circle. Draw a chord 6 cm. long in the circle which is parallel to XY.

29. Draw a circle with radius 1.7" and draw any diameter POQ. Find a point A in POQ produced from which the tangents drawn to the circle are 1 inch long. Measure QA.

30. Draw two circles with radii 3 cm. and 5 cm., the distance between their centres being 10 cm. Find a point P from which the tangents to the circles are both 2.5 cm. long. Draw the tangents from P and measure them.

PROBLEM 9.

To construct (i) the inscribed circle, (ii) one of the escribed circles of a given triangle.

Given a $\triangle ABC$.

To construct

- (i) the inscribed circle;
- (ii) the escribed circle touching BC and AB, AC produced.

(i) Constr. Bisect $\angle s$ B and C by lines BI and CI which meet in I ; draw $ID \perp BC$.

Then the circle drawn with I as centre and ID as radius is the req^d inscribed circle.

Proof. For, if IE, IF be drawn $\perp CA, AB$, since I lies on the bisector of $\angle ABC$,

$$ID = IF,$$

$$\text{Simly} \quad ID = IE$$

$$\therefore ID = IE = IF;$$

$\therefore$ the circle drawn with I as centre and ID as radius passes through E and F , and it touches the sides of the $\triangle ABC$. ($\angle s$ at D, E, F rt. $\angle s$)

(ii) Constr. Produce AB, AC to X, Y . Bisect $\angle s$ CBX, BCY by lines BI_1, CI_1 which meet in I_1 ;

draw $I_1D_1 \perp BC$.

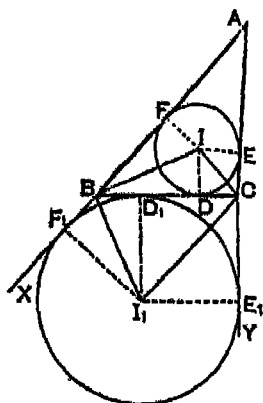
Then the circle drawn with I_1 as centre and I_1D_1 as radius is the req^d escribed circle.

Proof. For if I_1E_1, I_1F_1 be drawn $\perp AY, AX$, we have I_1D_1 equal to both I_1E_1 and I_1F_1 ,

$\therefore$ the circle passes through E_1 and F_1 ,

and as before it touches BC and AB, AC produced.

Q.E.F.



There are evidently three escribed circles (or ex-circles) in connection with a triangle. Their centres are called the ex-centres of the triangle.

THE CIRCLES ASSOCIATED WITH A TRIANGLE.

EXAMPLES 61.

It is very difficult to draw an inscribed (or escribed) circle of a triangle at all accurately. Use large arcs in bisecting angles; this helps to make the centre of the circle more accurate in position.

1. Draw triangles with the dimensions given below and construct the circles mentioned: then measure their radii:—

- (i) Sides 1.8", 2", 2.4"; incircle.
- (ii) Sides 1.5", 1.7", 2.2"; ex-circle touching 1.5" externally.
- (iii) Two sides 1.6" each, included angle a right angle; in-circle and one of the smaller ex-circles.
- (iv) Sides 1.5", 1.8", 1.9"; circumcircle and incircle.
- (v) Sides 8 cm., 10 cm., 12 cm.; circumcircles of given triangle and of the triangle formed by joining the middle points of its sides. What connection is there between the two radii?

2. Calculate the circum-radius of a Δ whose sides are all 2".
3. Calculate the radius of incircle of a Δ whose sides are all 4 cm.
4. If an equilateral triangle is inscribed in a circle of radius 1", find the length of each side.
5. If I is incentre of ΔABC , prove that $\angle BIC = 90^\circ + \frac{1}{2} \angle A$.
6. In the figure of Problem 9, prove that IB_1C is cyclic.
7. Prove that if the circumcentre of ΔABC is on the bisector of $\angle A$, $AB = AC$.
8. If the incentre and circumcentre of a triangle are the same point, prove that the triangle is equilateral.
- In the figure to Problem 9, prove that—
9. $BD - DC = AB - AC$.
10. $2 BD = AB + BC - CA$.
11. $BD_1 = \frac{1}{2}(a + b - c)$.
12. $AF_1 = \frac{1}{2}(BC + CA + AB)$.
13. The angles of ΔDEF are $90^\circ - \frac{A}{2}$, $90^\circ - \frac{B}{2}$, $90^\circ - \frac{C}{2}$.
14. If the incircle of a triangle touches one of its sides at the middle point, the other two sides are equal.
15. If in the figure to Problem 9, D and D_1 coincide, $AB = AC$.
16. Circles can be drawn with centres at the corners of any triangle to touch each other in pairs.

(SHORTLY.)

THE TOUCHING OF CIRCLES.

Two circles are said to touch each other at a point when they both touch the same line at that point. They will meet in that one place only (i.e. in two coincident points there) and the line will be a common tangent to the two circles at the point of contact.

If the circles are on *opposite sides* of the line, they are said to touch externally or to have external contact

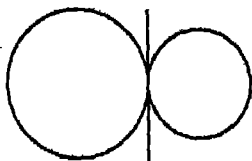


FIG. I.

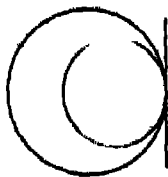


FIG. II.

FIG. 204.

(Fig. i), and if they are on the *same side* of the line, they are said to touch internally or to have internal contact (Fig. ii).

Note that if two circles touch at A, they will have a common tangent at A.

We could have looked at the question of two circles touching by considering first of all two circles cutting (say at A, A'), as in Fig. 205. Now if one circle is regarded as moving away from the other, the points A, A' ultimately coincide, the circles touch externally, and the common secant (or chord) AA' becomes the common tangent at A.

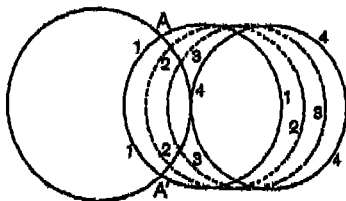


FIG. 205.

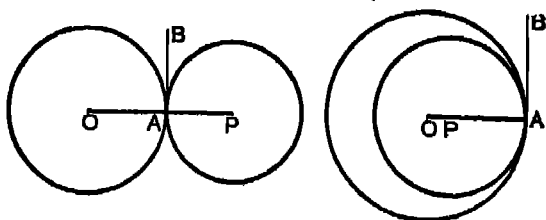
If the moving circle had moved the other way, the points A, A' would again have coincided, and we should have had internal contact.

When two circles meet but do not touch, they must of course cut each other. In this case the tangents to the circles at each point of intersection are distinct lines. The angle between the tangents at either point is said to be the angle at which the circles intersect.

THEOREM 48.

If two circles touch one another, their centres and the point of contact lie on a straight line.

Given two circles, centres O and P , touching at A .
To prove that O, A, P lie on a straight line.



Constr. Join OA, AP .

Proof. Since the circles touch at A , they have a common tangent BA at A .

Then, since AB is a tangent to the circle with centre O ,
 OA is perpendicular to AB . (*tgt. $\perp$ rad.*)

Similarly AP is also perpendicular to AB .

But there is only one straight line through A perp^r to AB ,
 $\therefore O, A, P$ lie on this straight line.

Q.E.D.

The above Theorem could be stated thus:—

If two circles touch externally, the distance between the centres is equal to the sum of the radii; and if two circles touch internally, the distance between the centres is equal to the difference of the radii.

The converse statements are also true. How then can we test whether two circles touch each other (i) externally, (ii) internally, if we know their radii and the distance between their centres?

THE CONTACT OF CIRCLES.

EXAMPLES 62.

NUMERICAL.

1. Two circles are drawn with radii $1\frac{7}{8}$ " and $2\frac{5}{8}$ "; if the distance between their centres is $4\frac{1}{2}$ ", why do they touch and what sort of contact have they? If the same circles are drawn with the distance between their centres as 8 ", why do they touch and what sort of contact have they?

2. Two circles (centres O, P) are drawn with radii 8 cm. and 5 cm., the distance between their centres being 13 cm. Show that a circle can be drawn with its centre 9.5 cm. from O and 8.5 cm. from P to touch both circles externally. What is its radius?

3. In the $\triangle ABC$ $a = 7$ cm., $b = 9$ cm., $c = 10.5$ cm. A circle is drawn with centre A and radius 6 cm. What is the radius of the circle drawn with centre B to have (i) external contact; (ii) internal contact with the first circle?

4. In the $\triangle ABC$ $a = 7.7$ cm., $b = 7.1$ cm., $c = 8$ cm.; show that circles drawn with A, B, C as centres and with radii 2.7 cm., 3.3 cm., 4.4 cm. will touch each other externally.

PRACTICAL.

Ex. 1-4 can all be done practically.

THEORETICAL.

5. Two circles touch internally at A, and T is any point on the common tangent at A. Prove that the other tangents from T, one to each circle, are equal.

6. Two circles touch externally at A and PAQ is a line through A meeting one circle in P and the other in Q. Prove that the radii through P and Q are parallel.

7. Two circles touch a given line AB at A and a line is drawn through A to meet the circles in X, Y. Prove that the tangents at X, Y are parallel.

8. Two circles touch externally at A and another line BC touches one circle at B and the other at C. Prove that $\angle BAC$ is a right $\angle$.

9. A variable circle with centre P touches externally two fixed circles with centres at A and B. Prove that $PA \sim PB$ is constant.

CONSTRUCTION OF CIRCLES.

There are many cases in which circles need to be constructed to satisfy given conditions, and it will be necessary for the student to be familiar with certain well-known loci in connection with circles. The following are very important.

If the locus cannot be seen at a glance by means of either a mental picture or a freehand figure, help may be obtained by taking the given circles and lines of definite sizes and drawing 5 or 6 circles in order to see what the locus *seems* to be. But it is then necessary for the student to make quite sure on theoretical grounds that this locus is the correct one. It is an excellent thing if practical illustrations of these loci are sought for in everyday life, e.g. one cog-wheel rolling on the outside of another whose centre is fixed, the wheel of a wheelbarrow moving along the ground, a soap-bubble increasing in size as it rests on the rim of the mouth of a clay pipe, and so on. (Which of the following do these illustrations refer to?)

State the locus of the centres of circles which

1. *Pass through two given points.*
2. *Touch two intersecting lines.*
3. *Touch two parallel lines.*
4. *Touch a given straight line at a given point.*
5. *Touch a given circle at a given point.*
6. *Pass through a given point and have a given radius.*
7. *Touch a given straight line and have a given radius.*
8. *Touch a given circle and have a given radius.*

These loci are constantly used in problems which have to do with the construction of circles. In such problems in the past we have always had given us the same number of conditions (*how many? give examples*), and the number of facts given us to enable the circle to be drawn is always the same—three.

If we omit one of the three facts we at once get an infinite number of possible circles; and we can generally find out from the list of loci given above what is the locus of their centres. By omitting another of the three given facts we try to find a second locus for the centre of the circle. (Label these loci when drawn as Locus I and Locus II). The centre of the required circle is the point (or points) of intersection of Locus I and Locus II.

PROBLEMS ON THE DRAWING OF CIRCLES.

Ex. 1. Draw a circle whose radius is $1.1''$ and find a point A which is $2.2''$ from the centre O. Now construct a circle to touch the given circle, to pass through A, and to have a radius of $.7''$.

We cannot omit the fact that the radius is $.7''$, as we do not know the locus of the centre of a circle which touches a circle and passes through a point outside it. We proceed thus:—

	∴ the locus of its centre is	This is
The circle	a concentric circle	Locus I;
(i) touches the given circle ext ^{ly} and has a radius of $.7''$ (No. 8 above),	with a radius $1.1'' + .7'' = 1.8''$,	
(ii) passes through A and has a radius of $.7''$ (No. 8 above).	a circle, centre A, and radius $.7''$.	Locus II.

Locus I meets Locus II in two points P and Q; if we take either of these points as centre and draw a circle with radius $.7''$ it will satisfy the required conditions.

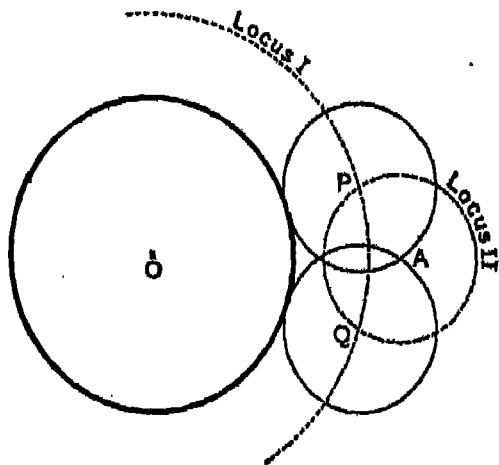


FIG. 206.

Ex. 2. Investigate the problem of constructing a circle to touch a given circle, centre O , at a given point A and to pass through another point B . Is the construction always possible?

Omit one of the three conditions at a time, as before:—

		$\therefore$ the locus of its	
The circle		centre is	This is
(i) touches the given		the diameter OA pro-	Locus I,
circle at A (No. 5		duced,	
above),			
(ii) passes through		the perp ^r bisector of	Locus II.
A and B (No. 1		AB .	
above).			

The centre of the circle required is the point C of intersection of Locus I and Locus II. The radius required is CA (or CB).

The centre can always be found and the circle drawn except when Locus I and Locus II are parallel; this is the case when $\angle BAO$ is a right angle, i.e. when B lies on the tangent at A to the given circle (see Fig. ii).

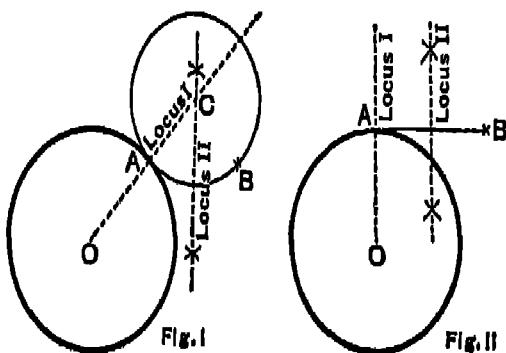


FIG. 207.

THE DRAWING OF CIRCLES.

EXAMPLES 63.

PRACTICAL.

1. AB is a straight line 2" long. Draw a circle of radius 1.4" to pass through A and B.

2. ABC is an angle of 80° , AB and BC being both 2 inches long. Draw a circle of radius .8" to touch AB and BC.

3. AB is a line 2 in. long. ABC is an angle of 70° and DAB is an angle of 110° , D and C being on the same side of AB, so that AD and BC are parallel. Draw a circle to touch the three lines DA, AB, and BC.

4. $\triangle ABC$ has $AB = 5$ cm., $BC = 7$ cm., $CA = 6$ cm. Draw a circle to touch BC at C and to pass through A. (Note that the circle must go through C.)

5. AB is a line 4.2" long and ACB is the arc of a semicircle drawn on AB as diameter. Draw a circle of radius .7" inside the semicircle to touch AB and the arc of the semicircle.

6. ABC is a triangle in which $AC = 2''$, $AB = 1.8''$, $BC = 1.1''$, and a circle is drawn with centre A and radius AB. Now draw a circle to pass through C and to touch the circle at B. Measure its radius.

7. A straight line XY is drawn and a point O taken which is 4 cm. away from XY. Draw a circle of radius $2\frac{1}{2}$ cm. to pass through O and to touch XY. How many such circles are there?

8. Draw a circle whose radius is 2" and a line .9" away from the centre. Draw a circle to touch the above circle and the above line and to have a radius of .5". How many such circles are there?

9. Two circles are drawn with radii 8 cm. and 5 cm., the distance between their centres being 9.5 cm. Draw a circle of radius 1.5 cm. to touch both circles externally.

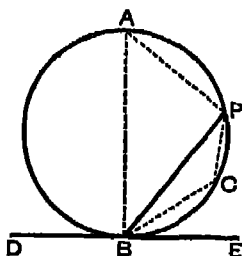
10. Two circles of radii 1.8" and 1.5" are drawn, their centres being 2" apart. Now draw a circle of radius .5" to touch the given circles internally.

11. In $\triangle ABC$, $a = 1.9''$, $b = 2.5''$, $c = 3.4''$. Draw the semicircle on AB as diameter on the same side of AB as C. Construct a circle of radius .7" to touch the arc of the semicircle internally and to pass through C.

THEOREM 49.

The angles which a tangent to a circle makes with a chord through the point of contact are equal to the angles in the alternate segments of the circle.

Given a circle ABC and a tangent DBE touching it at B, also a chord BP through B.



To prove that—

- (i) $\angle PBE = \angle$ in the alt. segment BAP;
and (ii) $\angle PBD = \angle$ in the alt. segment BCP.

Constr. Draw BA, the diameter of the circle passing through B; join PA. Take any point C in the minor arc BP; join PC, CB.

Proof. (i) BA is a diameter, (constr.)
 $\therefore \angle APB$ is a right $\angle$, ($\angle$ in a semicircle)
 $\therefore \angle s$ ABP and A are complementary, ($\angle s$ of a Δ)
 But $\angle ABE$ is a right $\angle$, (tgt. $\perp$ rad.)
 $\therefore \angle s$ ABP and PBE are complementary,
 $\therefore \angle PBE = \angle A$. (comps. of same $\angle$)
 $\therefore \angle PBE = \text{any } \angle$ in the segment BAP.

(ii) APCB is a cyclic quad^l,
 $\therefore \angle s$ A and C are supplementary, (*opp. $\angle s$*)
 Again, $\angle s$ PBE and PBD are supplementary, (*str. $\angle$*)
 But $\angle PBE = \angle A$, (proved)
 $\therefore$ their supplements are equal,
 i.e. $\angle PBD = \angle C$.

Q.E.D.

The converse of Theorem 49 is also important:

If the angle between a chord of a circle and any line through one end of the chord is equal to the angle in the alternate segment, that line will be a tangent to the circle.

Thus if $\angle BAC =$ the $\angle$ in the alternate segment BDA , AC will touch the circle at A .

The proof of this, by *Reductio ad Absurdum*, is left as an exercise for the student.

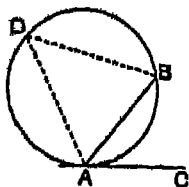


FIG. 217.

ALTERNATE SEGMENT.

EXAMPLES 65.

NUMERICAL.

1. ABC is a triangle with $\angle A = 50^\circ$; calculate the angle between the tangents at B and C to the circle ABC .

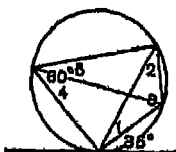


FIG. 220.

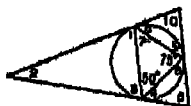


FIG. 221.

2. Calculate the size of the angles 1, 2, 3, 4, 5 in Fig. 220.
3. Calculate the size of the angles 1, 2, 3, . . . 10 in Fig. 221.
4. ABC, ADE are tangents to a circle BDF at B and D ; $\angle CBF = 72^\circ$, $\angle EDF = 61^\circ$; calculate $\angle BFD$ and $\angle BAD$.
5. Two of the angles of a cyclic quadrilateral are 76° and 102° . Find the angle between the tangents to the circle at the first vertex and the opposite one.
6. An obtuse angle ABC in a segment of a circle is a° ; prove that the angle between the tangents at A and C is $(2a - 180)^\circ$.

THEORETICAL.

7. Use Theorem 49 to prove that tangents to a circle from an external point are equal.
8. If an equilateral triangle is inscribed in a circle, prove that the tangents at its vertices form a Δ which is also equilateral.
9. AB touches a circle at A , and AX is a chord of the circle; if XY is a chord $\parallel AB$, prove that chord $AX =$ chord AY .

10. Two circles meet in A, B , and AC, AD are the tangents at A to the two circles, meeting the other circles in C, D ; prove that $\angle ABC = \angle ABD$.

11. AB and CD are parallel chords of a circle; the tangent at A meets DC produced in X , and BD is produced to E . Prove that $\angle XAD = \angle EDC$.

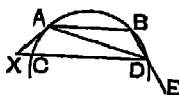


Fig. 222.

12. In $\triangle ABC$ the tangent at C to the circle ABC meets AB produced in D ; if the bisector of $\angle ACB$ meets AB in E , prove that $DE = DC$.

13. $ABCD$ is a cyclic quadrilateral, whose diagonals meet in O . Prove that, if a circle be drawn through A, B, O , the tangent to this circle at O is parallel to CD .

14. $ABCD$ is a cyclic quadrilateral; a circle is drawn to touch AB at A and to pass through D , cutting CD again in E ; prove that $AE \parallel BC$.

15. $ABCD$ is a cyclic quadrilateral, and the tangents at A and C to the circle $ABCD$ meet in E . If the arc ABC is a minor arc, prove that $\angle E = \angle ABC - \angle ADC$.

16. Two circles touch internally at A , and two lines are drawn through A meeting one circle in X and P and the other in Y and Q . Prove that $XP \parallel QY$.

17. Prove Ex. 16 when the circles have external contact.

18. $ABCD$ is a trapezium, and the non-parallel sides AD, BC meet when produced at O . Prove that the circles OAB, OCD touch at O .

19. Two circles touch internally at A ; BC is a chord of the larger circle touching the smaller at D . Prove that AD bisects $\angle BAC$.

20. Two circles touch internally at A ; BC is a chord of the outer circle cutting the inner at D and E . Prove that $\angle BAD = \angle CAE$.

21. OAB, OCD are straight lines meeting in O . Prove that the acute angle between the tangents at O to the circles OAC, OBD is equal to the acute angle between the lines AC, BD .

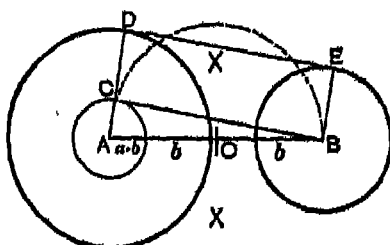
22. D is a point in the base BC of $\triangle ABC$; two circles are drawn to pass through D , one of them touching AB at B , and the other touching AC at C . If the circles meet again in E , prove that A, B, E, C are concyclic points.

23. Two circles ABC, ADE touch externally and a straight line $BCDE$ is drawn to cut them both; prove that BE and CD subtend supplementary angles at A .

COMMON TANGENTS.

PROBLEM 10A.

To draw an external common tangent to two circles.



- Given two circles with centres A, B, and radii a , b , where $a > b$.

To draw an external common tangent to the circles.

Constr. Bisect AB at O, and on AB as diameter construct a semicircle; with centre A and radius $a - b$ (the difference of the radii) draw a circle, cutting the semicircle in C.

Join AC and produce it to meet the large circle in D: join CB.

* Draw $BE \parallel AD$ to meet the circle of radius b in E; join DE.

Then DE is the required external common tangent.

Proof. Since $AD = a$ and $AC = a - b$, $\therefore CD = b = BE$;
also $CD \parallel BE$, (constr.)

$\therefore CDEB$ is a par^m. (CD and $BE =$ and $\parallel$)

Also $\angle ACB$ is a right $\angle$, ($\angle$ in a semicircle)

$\therefore DCB$ is a right $\angle$, (str. $\angle$ at C)

$\therefore CE$ is a rectangle,

and $\therefore \angle$ s at D and E are right $\angle$ s,

so that DE touches both circles, (tgt. $\perp$ rad.)

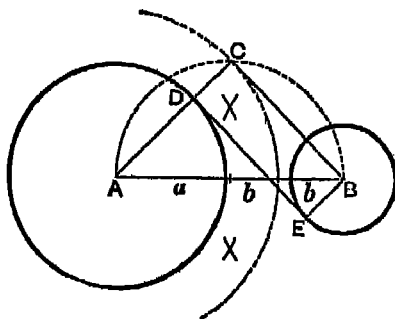
i.e. DE is an ext. common tangent.

Q.E.F.

* This can be done with a set square, but a simple method, using compasses, is to draw an arc with centre D and radius CB, cutting the circle of radius b in E.

PROBLEM 10B.

To draw an internal common tangent to two circles.



Given two circles with centres A and B and radii a and b .

To draw an internal common tangent to the given circles.

Constr. Bisect AB at O and on AB as diameter construct a semicircle; with centre A and radius $a + b$ (the sum of the radii) draw a circle, cutting the semicircle in C .

Join AC , cutting the circle with radius a in D ; join CB .

*Draw BE par^l to CA to meet circle, radius b , in E ; join DE .

Then shall DE be a common internal tangent.

Proof. $DC = AC - AD = (a + b) - a = b = BE$,

also $DC \parallel BE$,

$\therefore DCBE$ is a par^m. (DC and $EB =$ and $\parallel$)

But $\angle C$ is a right $\angle$, ($\angle$ in a semicircle)

$\therefore DB$ is a rectangle,

and $\therefore CDE$ and DEB are right $\angle$ s,

so that DE touches both circles, ($tgt. \perp rad.$)

i.e. DE is an internal common tangent.

Q.E.F.

* Either by means of a set square, or by drawing an arc with centre D and radius CB , cutting the circle of radius b in E .

COMMON TANGENTS.

EXAMPLES 66.

NUMERICAL AND PRACTICAL.

1. Draw two circles with radii 1.5'' and .8'', with their centres 2.5'' apart; draw an external common tangent, and measure its length. Check by means of Pythagoras.

2. Draw two circles as in Ex. 1 and draw an internal common tangent. Find its length (a) by measurement, (b) by calculation.

3. Draw an external common tangent to two circles, such that

$$(i) \ a = 7 \text{ cm.}, \ b = 4 \text{ cm.},$$

$$D \text{ (distance between centres)} = 9 \text{ cm.}$$

$$(ii) \ a = 2.5'', \ b = 1'', \ D = 3.5''.$$

In each case find the length of the common tangent in two ways.

4. Draw an internal common tangent to two circles, such that

$$(i) \ a = 3 \text{ cm.}, \ b = 4 \text{ cm.}, \ D = 9 \text{ cm.}$$

$$(ii) \ a = 1.7'', \ b = .7'', \ D = 3''.$$

Measure and calculate its length in each case.

5. Two circles have radii 5 cm. and 8 cm., their centres being 15 cm. apart. Calculate the length of an internal and of an external common tangent. Which is the greater?

6. If circles have radii a and b and the distance between their centres is D , prove that, if T and T' are the lengths of their external and internal common tangents,

$$T^2 = D^2 - (a - b)^2 \text{ and } T'^2 = D^2 - (a + b)^2$$

Which is the greater, T or T' ?

7. Prove in Ex. 6 that $T^2 - T'^2 = 4ab$.

8. If T is the length of an external common tangent to two circles of radii a and b which touch externally, prove by Pythagoras that $T^2 = 4ab$.

9. Two circles touch externally at A . State a construction for finding a point in the common tangent at A such that the other tangents from it to the two circles are in a straight line.

10. If two circles of radii a and $2a$ touch externally at A , and the common tangent at A meets at B an external common tangent; prove that $AB = a\sqrt{2}$.

11. The smaller of two circles lies entirely within the larger. Show how to draw a chord of the larger to touch the smaller and to have a given length.

PROBLEM 11.

On a given straight line as base, to construct a segment of a circle which shall contain an angle equal to a given angle.

Given a line AB and an $\angle P$.

To construct on AB as base a segment of a circle containing an $\angle = P$.

Constr. Make an angle $BAC = \angle P$;

draw $AD \perp AC$.

Draw perp^r bisector of AB , meeting AD in O .

With centre O and radius OA draw an arc of a circle on the opposite side of AB to C .

Since O is on the perp^r bisector of AB ,

$\therefore OA = OB$, $\therefore$ the circle goes through B .

Then shall segment AEB be the segment required.

Proof

$AC \perp$ radius OA ,

$\therefore AC$ touches the circle,

$\therefore \angle CAB = \angle$ in alt. segment AEB .

But $\angle CAB = \angle P$,

$\therefore \angle$ in segment $AEB = \angle P$.

Q.E.F.

Note 1. If in the above problem $\angle P$ is given in degrees (say 64°), the line AC need not be drawn; for $\angle AOB = 128^\circ$ (why?), and $\angle s$ OAB and OBA can be found (what are they?), and if these angles are drawn, O is obtained, and the circle can be drawn. (See fig. 223.)

Note 2. If a segment has to be drawn on a given base AB containing a given obtuse angle (say 115°), the angle AOB at the centre O is reflex (230°), $\therefore \angle AOB$ in $\triangle AOB$ is 130° $\therefore \angle s$ ABO and BAO are both 25° . Hence O can be found and the circle drawn.

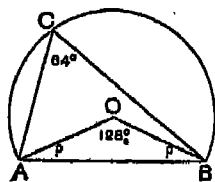
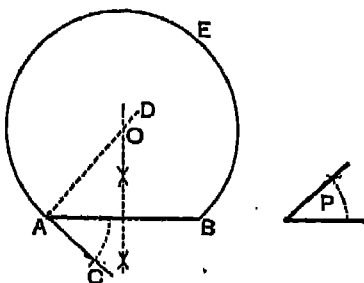


FIG. 223.

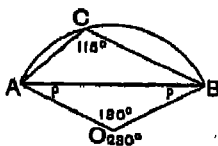


FIG. 224.

Note 3. The method of intersection of loci can often be used in connection with this construction.

Ex. Draw a triangle ABC, given $BC = 5.6$ cm., $\angle A = 55^\circ$, altitude $AD = 3$ cm.

Draw the base BC.

(i) Then knowing the vertical $\angle A$, the locus of A is an arc of a circle through B and C. (Locus I.)

(ii) Knowing the altitude, the locus of A is a line $\parallel BC$. (Locus II.)

We can now find the position of A, viz. where these loci meet. Two easy and important constructions follow.

TO INSCRIBE IN A GIVEN CIRCLE A TRIANGLE ABC WHOSE ANGLES ARE GIVEN (SAY $50^\circ, 60^\circ, 70^\circ$).

Since $\angle A$ is to be 50° , the base BC will subtend an angle of 100° at the centre O. So also CA will subtend an angle of 120° at O.

Hence we draw a radius OC and make on opposite sides of it $\angle BOC = 100^\circ$ and $\angle COA = 120^\circ$.

$\therefore \angle BAC$ will be $\frac{1}{2} \angle BOC$, i.e. 50° , and $\angle CBA$ will be 60° , as required.

$\therefore$ the third $\angle ACB$ must be 70° , as required.

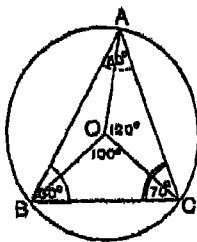


Fig. 225.

TO CIRCUMSCRIBE ABOUT A GIVEN CIRCLE A TRIANGLE WHOSE ANGLES ARE GIVEN.

Analysis. If ABC is the triangle when drawn, join O, the centre of the circle, to D, E, F, the points of contact of the sides. If it is known that $\angle A = 60^\circ$, $\angle B = 50^\circ$, $\angle C = 70^\circ$, can the size of the three angles at O be found? [Consider quadrilateral BDOF; how many angles in it are known? Find the size of the fourth angle.]

Evidently $\angle FOD = 180^\circ - B = 130^\circ$, $\angle DOE = 180^\circ - C = 110^\circ$.

Hence the construction is:

Draw a radius OD, make $\angle DOF = 180^\circ - B$, and $\angle DOE = 180^\circ - C$; now draw tangents at D, E, F in order to form a triangle ABC. Prove that $\angle B = 50^\circ$, $\angle C = 70^\circ$, $\angle A = 60^\circ$.

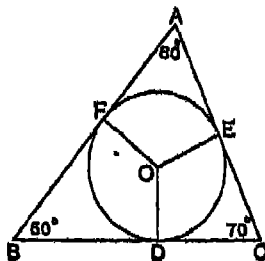


Fig. 226.

EXAMPLES 67.

PRACTICAL.

1. Draw a line BC representing a piece of coast-line, two miles long, and running E. and W., and draw the curve showing the positions of all ships (on the north side of BC) at which BC subtends an angle of (i) 70° ; (ii) 90° ; (iii) 105° .

2. Draw a circle of radius $1.5''$ and inscribe in it a triangle whose angles are 55° , 60° , 65° . Measure the longest side.

3. A ladder AB 14 feet long rests with one end A on the ground and the other end B against a vertical wall. If the lower end slips and B remains on the wall, plot the locus of the middle point of the ladder.

4. Plot the locus of the middle points of a series of chords of a circle which all pass through a fixed point O inside the circle.

Construct triangles from the data in questions 5-8.

5. Base = $2''$, vertical $\angle = 50^\circ$, altitude = $1.8''$. Measure the sides.

6. Base = $1.5''$, vertical $\angle = 40^\circ$, one other side = $1''$. Measure the other side.

7. Base = 6 cm., vertical $\angle = 80^\circ$, median bisecting base = 3.5 cm.

8. Base = $3.1''$, vertical $\angle = 110^\circ$, altitude = $.8''$.

9. A, B, C are three lighthouses in order on a straight coast-line running E. and W., AB being $1\frac{1}{2}$ miles long and BC $\frac{1}{2}$ mile. A shipwrecked sailor on a rock R notices that B is due south of him and that the angle ARC is 70° . Draw a figure showing his position and measure his distance from A.

10. ABC is a triangle in which AO = 7 cm., BO = 10 cm., AB = 5 cm. Construct the position of a point P such that $\angle BPA = 30^\circ$, $\angle CPA = 30^\circ$, $\angle BPC = 60^\circ$. Measure PA.

11. Construct a parallelogram in which the diagonals are 7 cm. and 5.5 cm. and the acute angles of the figure are 64° . Measure the sides of the figure.

12. Construct a parallelogram, given that one side = $2.8''$, the angle subtended by that side at the meet of the diagonals is 118° , and one diagonal is $3.4''$. Measure the other side.

REGULAR FIGURES INSIDE AND OUTSIDE CIRCLES.

INSCRIBED REGULAR POLYGONS.

If we divide the angle at the centre of the circle into equal angles $AOB, BOC, COD, \dots$ at the centre of a circle and join the points $A, B, C, \dots$ by chords $AB, BC, \dots$, we can prove that—

$$(i) \quad AB = BC = CD = \dots$$

$$(ii) \quad \angle ABC = \angle BCD = \dots$$

Hence the figure $ABCD \dots$ has all its sides equal and all its angles equal, i.e. it is a regular polygon.

Thus we see that, in order to inscribe a regular polygon of n sides in a given circle, we draw at the centre n equal angles, each being $\frac{360^\circ}{n}$. Then the ends of the adjacent radii have to be joined up.

In most cases the angles have to be made by means of a protractor; but in a few cases compasses will serve.

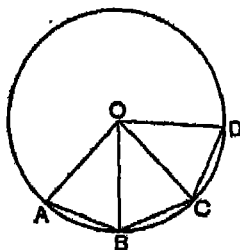


FIG. 227.

PROBLEM 12.

To inscribe in a given circle a regular figure of 3, 4, 6 or 8 sides.

(i) AN EQUILATERAL TRIANGLE.

We make angles of 120° at the centre of the circle.

This can be done by making 60° at the centre twice, i.e. by taking a radius equal to the radius of the circle twice over round the circumference.

We thus get $\triangle ACE$ in Fig. 228.

(ii) A SQUARE.

We make angles of 90° at the centre of the circle. Thus we draw an inscribed square by making two diameters at right angles and joining their extremities.

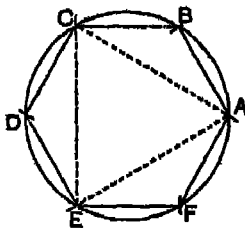


FIG. 228.

(iii) A REGULAR HEXAGON.

We make angles of 60° at the centre of the circle.

We thus get the hexagon $ABCDEF$ in Fig. 228.

(iv) A REGULAR OCTAGON.

We now make angles of 45° at the centre of the circle.

Thus we draw two $\perp$ diameters and bisect the angles between them.

CIRCUMSCRIBED REGULAR POLYGONS.

If we make n equal angles $AOB, BOC, \dots$ at the centre of a circle (each equal to $\frac{360^\circ}{n}$), and tangents are drawn to the circle

at $A, B, C, \dots$, a circumscribed polygon will be obtained, which can easily be proved to be equilateral and equiangular (i.e. regular).

[Fig. 229 shows two regular pentagons, one inscribed in a circle, the other circumscribed about the same circle.]

Thus we can circumscribe a regular polygon of n sides about a given circle, and in the cases when $n = 3, 4, 6$ or 8 , it can be done without a protractor, if need be. [See Problem 12 above.]

Note that the centre of the circle is also the centre of any inscribed (or circumscribed) polygon; for it can be proved to be the point of concurrence of the bisectors of the angles and the perp^r bisectors of the sides of any such polygon. This indicates the method of inscribing a circle in (or circumscribing a circle about) a regular polygon.

The centre of the circle is obtained *either* by bisecting two adjacent angles *or* by drawing the perp^r bisectors of two adjacent sides of the polygon.

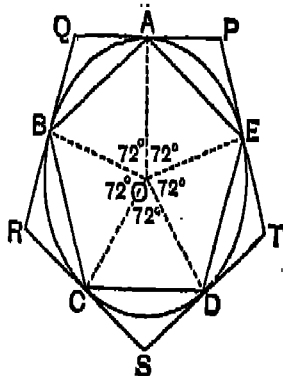


Fig. 229.

EXAMPLES 68.

PRACTICAL.

1. Draw a circle of radius $1\frac{1}{8}$ " and inscribe a square in it. Measure one side.

2. Draw a square circumscribed about a given circle of radius 2 "; measure one side.

3. Draw in separate figures, in connection with a circle of radius $1\frac{1}{2}$ ", an inscribed and a circumscribed

(i) equilateral triangle; (ii) hexagon; (iii) octagon.

Measure a side of each of the six figures.

4. Draw a regular pentagon inside a circle of radius 4 cm. By means of perpendicular bisectors of sides (*how many sides?*) inscribe a circle in the pentagon, verifying that the in-centre and circum-centre are the same point. Measure the radius of the second circle.

5. Circumscribe a hexagon symmetrically about a given hexagon of side $1\frac{1}{8}$ ". How long are its sides?

6. A square is drawn with sides $2\frac{1}{2}$ " long. Cut off equal triangles from each of its corners so that the figure left is a regular octagon. Measure a side of this figure.

7. Inside an equilateral triangle (side 2 ") draw (as in Fig. 230) three circles to touch each other in pairs, each circle also touching two sides of the triangle. Measure a diameter of one circle.

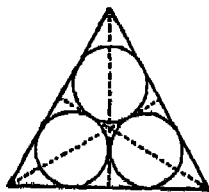


FIG. 230.

THEORETICAL.

8. AB and AC (on opposite sides of A) are sides of regular polygons of 10 and 15 sides inscribed in a given circle ABC. If BC is joined, prove that BC is a side of a regular hexagon inscribed in the circle.

9. If A is a point on a circle and AB and AC (on the same side of A) are sides of inscribed regular polygons of 10 and 12 sides, show that BC, when joined, is a side of another inscribed regular polygon. How many sides has it?

10. If ABCDE is a regular pentagon, prove that CA and CE trisect the angle C.

11. If ABCDE is a regular pentagon, and O its circum-centre, and if AB, DC produced meet in F, show that A, F, C, O are cyclic.

CIRCUMFERENCE OF A CIRCLE.

The pupil may have already learnt of the relationship between the circumference of a circle and its diameter. The result is that

$$\frac{\text{Circumference}}{\text{Diameter}} \text{ is the same for all circles.}$$

The ratio cannot be expressed exactly, i.e. it is said to be an incommensurable quantity. Its value to four decimal places is 3.1416. For elementary work $3\frac{1}{4}$ or 3.14 can be used. When the actual value of the ratio is not wanted it is represented by the Greek letter π (pronounced 'pi').

$$\begin{aligned} \text{Thus } \text{Circumference of a circle} &= \pi \times \text{Diameter,} \\ &= 2\pi r, r \text{ being the radius.} \end{aligned}$$

SUMMARY OF METHODS OF VERIFYING PRACTICALLY THAT

$$\frac{\text{Circumference}}{\text{Diameter}} = \text{constant.}$$

The average of the answers obtained by the class should work out at 3.14.

METHOD I. A narrow strip of paper is wound round a cylinder once. A pin-hole is then pricked through the overlapping ends. Thus, when the paper is unwrapped, the distance between the pin-holes is the length of the circumference. The diameter of the cylinder can be obtained *either* directly by means of a ruler or by laying it flat between two wooden blocks which are kept straight by the edges being kept flat against a ruler. The ruler is then used to measure the gap between the wooden blocks, and this gives us the diameter of the cylinder.

METHOD II. A variation of the previous method consists in winding a thin piece of string round the cylinder ten or more times, taking care that the various layers of string touch, and do not cross, one another. After cutting off the string so that the two ends are on a line parallel to the axis (a ruler is useful for this), the string can be unrolled and measured, and so ten (or more) circumferences can be found. After finding the length of the circumference by division, proceed as in Method I.

METHOD III. Unequal circles are drawn on paper, and their circumferences found by means of a pair of dividers stretched apart a short distance so that the arc and chord for this distance are practically the same. The number of times this distance (which should be an exact number of millimetres or tenths of an inch) goes into the circumference and the small length of arc remaining are noted, and so the circumference is found. The diameter is measured directly.

METHOD IV. The fact that $\frac{\text{Circumference}}{\text{Diameter}}$ is the same for all circles can be verified easily, without finding the value of the fraction, by modifying Method III. Draw two circles, the radius of one being twice the radius of the other. Find how, by means of the dividers, many times a small length of arc (or chord) goes into the circumference of each circle, and show that it goes into the larger one twice as many times as into the other.

$$\text{Thus } \frac{\text{First Circumference}}{\text{Second Circumference}} = \frac{2}{1} = \frac{\text{First diameter}}{\text{Second diameter}}$$

THE AREA OF A CIRCLE.

To prove that the area of a circle of radius $r = \pi r^2$.

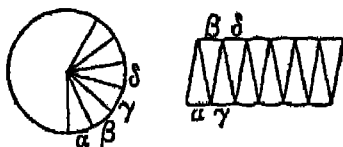


FIG. 231.

METHOD I. Divide the angle at the centre O into an even number ($2n$) of equal parts. Then the arcs $\alpha, \beta, \gamma, \dots$ that are cut off are all equal, and so are the sectors. Arrange the sectors as in the second figure, so that the central angles point alternately up and down. Then if n is large the second figure looks almost like a parallelogram, and ultimately, if n is taken large enough, it becomes a rectangle, whose height is r and whose base is half the circumference of the circle.

Thus, by taking n sufficiently large, we see that the area of the circle becomes equal to $\pi r \times r$ (base $\times$ ht.) $= \pi r^2$.

METHOD II. Make n equal angles at the centre, and by drawing tangents obtain a circumscribed regular polygon of n sides, viz. $ABCD \dots$

$$\begin{aligned} \text{Then area of polygon} &= n \times \text{area of } \triangle OAB \\ &= n \times \frac{1}{2} AB \times r \\ &= n \times \text{base } AB \times \frac{1}{2} r \\ &= \text{perimeter of polygon} \\ &\quad \times \frac{1}{2} r. \end{aligned}$$

Now suppose n becomes indefinitely large;

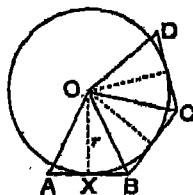


FIG. 232.

then the area of polygon becomes indistinguishable from the area of the circle, and the perimeter of polygon is identical with the circumference of the circle;

$$\begin{aligned}\therefore \text{area of circle} &= \text{circumference of circle} \times \frac{1}{2}r \\ &= 2\pi r \times \frac{1}{2}r = \pi r^2.\end{aligned}$$

ARCS AND SECTORS.

LENGTH OF AN ARC. Since equal angles AOB, BOC, COD, ...

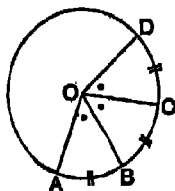


Fig. 233.

at the centre O of a circle stand on equal arcs AB, BC, CD, ...

it follows that if $\angle AOB = \frac{1}{n}$ th of 360° ,

arc AB is $\frac{1}{n}$ th of circumference.

$$\therefore \frac{\text{arc AB}}{\text{circumference}} = \frac{\text{angle AOB}}{360^\circ},$$

$$\therefore \text{if } \angle AOB = \theta, \frac{\text{arc AB}}{2\pi r} = \frac{\theta}{360^\circ}$$

$$\therefore \text{length of arc} = \frac{\pi r \theta}{180^\circ}.$$

AREA OF SECTOR. Since the sectors in Fig. 233 are all equal, it follows that

$$\frac{\text{area of sector AOB}}{\text{area of circle}} = \frac{\text{angle AOB}}{360^\circ},$$

$$\begin{aligned}\therefore \text{area of sector} &= \pi r^2 \times \frac{\theta}{360} = \frac{1}{2} \frac{\pi r \theta}{180} \times r \\ &= \frac{1}{2} \text{arc} \times \text{radius}.\end{aligned}$$

This truth can be remembered by noticing that a sector is a sort of triangle on a curved base, the height at every part being the radius, and the base being the arc.

CURVED SURFACES OF SOLIDS.

If we take a cylinder of height h and radius of base r , and cut the



FIG. 234.

curved surface by a line parallel to its height and then open it out into a rectangle, we see that

$$\begin{aligned} \therefore \text{the area of the curved surface of cylinder} \\ &= \text{base} \times \text{height} \\ &= 2\pi r \times h = 2\pi rh. \end{aligned}$$

In the case of a cone of slant height s and radius of base r , it can



FIG. 235.

be cut along the slant height and opened out into a sector of a circle, the arc being $2\pi r$ and the radius being s ,

$$\begin{aligned} \therefore \text{the area of the curved surface of a cone} \\ &= \text{area of sector} = \frac{1}{2} \text{arc} \times \text{radius} = \frac{1}{2} 2\pi r \times s \\ &= \pi rs. \end{aligned}$$

We shall assume that—

the area of the surface of a sphere = the area of the curved surface of a circumscribing cylinder (see figure 236);

$$\begin{aligned} \therefore \text{the area of the surface of a sphere} \\ &= 2\pi r \times 2r = 4\pi r^2. \end{aligned}$$



FIG. 236.

EXAMPLES 69.

Take π as $3\frac{1}{2}$.

1. Find the circumferences of circles whose radii are 4·9", 1·4", 4·5".
2. Find the radii of circles whose circumferences are 2·2", 9·8", 2·5" (correct to ·1 in.).
3. How far does a bicycle wheel 28" in diameter roll in one revolution?
4. A mile race is run on a circular track of radius 42 yards. How many times round must the *scratch* man run?
5. An arc of a circle of radius 5 ins. subtends an angle of 30° at the centre; find its length (correct to ·1 in.).
6. A sector of a circle of radius 3·5 cm. has a total perimeter of 11·4 cm. Find the angle between the two radii.
7. If a ship sails southwards along a meridian from Lat. 55° N. to Lat. 15° N., find how far it travels (to the nearest mile). (Take radius of earth as 3,960 miles.)
8. If a cart seven feet wide is driven round a circular track a quarter of a mile in circuit with its outer wheel on the boundary, how far will the inner wheel travel?
9. Find the areas of circles whose radii are 1 ft. 2 ins., 5·6 cm., and 15 yds. (to the nearest tenth of a sq. yd.).
10. Find the radii of circles whose areas are 2,464 sq. ins., 100 sq. ft., 54 sq. cm. (Answers correct to 3 figures.)
11. The angle of a sector of a circle (radius 7 cm.) is 72°. Find its area and the length of its arc.
12. Two sides of a field are straight fences meeting at O at an angle of 45°. If a cow be tethered to a stake in the ground at O by a rope 10 yards long, find the area of the field which it can graze (correct to ·1 sq. yd.).
13. Find the area between two concentric circles of radii 7" and 4".
14. A grass lawn in the shape of a circle has a radius of 200 feet, and there is a flower border 10 feet wide surrounding it. Find the area of the border (ignoring decimals of a square foot.)
15. Prove that the area of the band between two concentric circles of circumferences C and C' is $\frac{C + C'}{2} \times t$, where t is the thickness of the band.
16. Find the curved surface and total surface of a closed cylinder of height 5" and radius 7".

17. Find the area of the curved surface of a cone whose radius = 8.3" and whose slant height is 5".

18. Find the area of the curved surface of a cone whose radius is 4.2 cm. and whose perpendicular height is 5.6 cm.

19. A piece of tin in the form of a semicircle (radius 10 cm.) is bent so as to form a conical cap. Find the diameter of the cone.

20. How many times is the area of the surface of a sphere greater than the area of a circle having the same radius as the sphere?

21. What area of canvas will make a conical tent 8 feet high and 12 feet in diameter?

22. Find the area of a segment of a circle, radius 5", the length of the chord of the segment being 5". (A segment is a sector — a triangle.)

23. ABC is a right-angled triangle, C being the right angle, and semicircles are drawn on the three sides as diameters. Prove that the area of the semicircle drawn on AB = the sum of the areas of the other two semicircles.

24. Three halfpenny coins are placed on a table so that each touches the other two. Find the length of a piece of string which just surrounds them and is taut, the radius of a halfpenny being $\frac{1}{2}$ inch.

25. Two coins are placed in contact, the radius of the smaller being 2 cm. Find the length of a taut piece of string which just surrounds them, if the straight pieces of string are at right angles to each other.

26. If a regular hexagon be inscribed in a circle of radius 2", find the total area between the hexagon and the circumference.

27. In Fig. 237 semicircles are drawn, as shown, on the sides of a right-angled triangle as diameters. Show that the sum of the shaded areas is equal to the area of $\triangle ABC$.

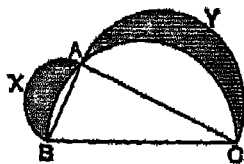


FIG. 237.

28. Fig. 238 shows a semicircle on AB and four other semicircles drawn on AX, XY, YZ, ZB as diameters. Show that the total perimeter of the first semicircle is equal to the sum of the perimeters of the other four, whatever the sizes of the latter.

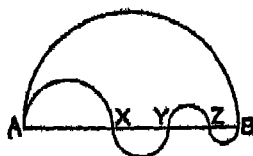


FIG. 238.

29. If in Fig. 238 $AX = YZ$ and $XY = ZB$, show that the area of the original semicircle cannot be equal to the sum of the four circles on AX, XY, YZ, ZB as diameters except in the special case in which two of the four segments of AB are zero and the other two are half the line.

SECTION V.

THE RECTANGLE.

A rectangle is said to be contained by two lines when those lines are respectively equal to two adjacent sides of the rectangle.

Thus the rectangle $ABCD$ is said to be contained by the lines AB , AD or by the lines PQ , XY (since $PQ = AB$ and $XY = AD$). It will often be referred to as rect. PQ , XY or as PQ , XY .

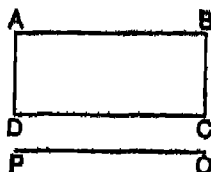


FIG. 247.

A line (AB) is said to be divided internally (at C) if the point of section (C) is on the line (AB) (see Fig. i).

A line (AB) is said to be divided externally (at C) if the point of section (C) is on the line (AB) produced (see Fig. ii).

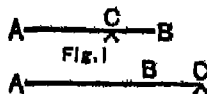


FIG. 248.

The segments (or parts) in each case are AC and CB .

We have already learnt that if a and b represent the number of units of length in two lines, then ab represents the number of units of area in the rectangle contained by the lines, and a^2 represents the number of units of area in the square drawn on the first line. Briefly we shall say that if a and b are the lengths of two lines, ab is the area contained by them, and a^2 is the area of the square on the line whose length is a .

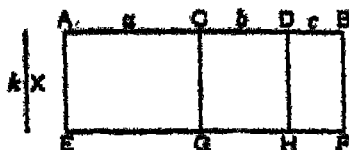
We are now able to illustrate several well-known algebraical identities by means of geometrical figures.

They are four in number:—

- | | |
|---------------------------|--------------------------|
| I. $k(a + b + c + \dots)$ | $= ka + kb + kc + \dots$ |
| II. $(a + b)^2$ | $= a^2 + b^2 + 2ab.$ |
| III. $(a - b)^2$ | $= a^2 + b^2 - 2ab.$ |
| IV. $a^2 - b^2$ | $= (a + b)(a - b).$ |

PROBLEM 13 I

To construct a figure to illustrate the identity
 $k(a + b + c) = ka + kb + kc$.



Given a line X of length k , and a line AB of length $a + b + c$, divided at C and D so that $AC = a$, $CD = b$, $DB = c$.

To show by means of a figure that

$$k(a + b + c) = ka + kb + kc.$$

Constr. On AB draw a rectangle $BAEF$ such that $AE = k$;
draw OG and $DH \parallel AE$.

Proof. Then AG , CH , DF are all rectangles.

Also fig. $AF = \text{fig. } AG + \text{fig. } CH + \text{fig. } DF$;

$$\therefore k(a + b + c) = ka + kb + kc$$

[i.e. $X \cdot AB = X \cdot AC + X \cdot CD + X \cdot DB$.]

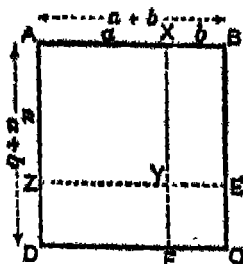
This last fact can be enunciated thus—

The rectangle contained by two lines, one of which is divided into several parts, is equal to the sum of the rectangles contained by the undivided line and the parts of the divided line in turn.

PROBLEM 13 II

To construct a figure to illustrate the identity

$$(a + b)^2 = a^2 + b^2 + 2ab.$$



Given a line AB of length $a + b$ divided (internally) at X so that $AX = a$ and $XB = b$.

To show by means of a figure that

$$(a + b)^2 = a^2 + b^2 + 2ab$$

Constr. On AB and AX draw squares ABCD, AXYZ;
produce ZY to meet BC in E,
and XY to meet DC in F.

Proof. Then $AZ = a$, $AD = a + b$, $\therefore ZD = b$; also $YE = b$.
 $\therefore$ YC is a square whose area = b^2 .

Also fig. AC = fig. AY + fig. YC + fig. DY + fig. BY,

$$\therefore (a + b)^2 = a^2 + b^2 + 2ab.$$

[i.e. $AB^2 = AX^2 + XB^2 + 2AX \cdot XB$.]

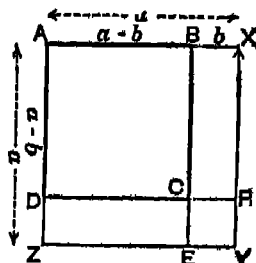
This fact can be enunciated thus:—

If a straight line be divided internally, the square on the line equals the sum of the squares on its segments, plus twice the rectangle contained by the segments.

PROBLEM 18 III

To construct a figure to illustrate the identity

$$(a - b)^2 = a^2 + b^2 - 2ab.$$



Given a line AB of length $a - b$ divided externally at X so that $AX = a$ and $XB = b$.

To show by means of a figure that

$$(a - b)^2 = a^2 + b^2 - 2ab.$$

Constr. On AB and AX draw squares ABCD, AXYZ;
produce BC to meet ZY in E,
and DC to meet XY in F.

Proof. Then $AZ = a$, $AD = a - b$, $\therefore DZ = b$; also $CF = b$.
 $\therefore$ CY is a square whose area is b^2 .

Also fig. AC = fig. AY + fig. CY - fig. DY - fig. BY,

$$\therefore (a - b)^2 = a^2 + b^2 - 2ab.$$

[i.e. $AB^2 = AX^2 + XB^2 - 2AX \cdot XB$.]

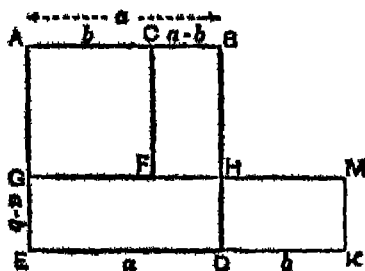
This fact can be enunciated thus:—

If a straight line be divided externally, the square on the line equals the sum of the squares on its segments, minus twice the rectangle contained by the segments.

PROBLEM 13 IV

To construct a figure to illustrate the identity

$$a^2 - b^2 = (a + b)(a - b).$$



Given two lines AB, AC of lengths a , b , of which $a > b$.

To show by means of a figure that

$$a^2 - b^2 = (a + b)(a - b).$$

Constr. Place AC so that it lies along AB, $\therefore CB = a - b$.

On AB and AC draw squares ABDE, ACFG;

produce GF to meet BD in H;

produce ED to K, making DK = b ;

complete the rectangle HDKM.

Proof. Then EK = $a + b$ and KM = EG = $a - b$:

also fig. CH = sq. HK.

(constr.)

Then $a^2 - b^2 =$ sq. AD - sq. AF,

= fig. CH + fig. GD,

= fig. HK + fig. GD,

= fig. GK,

= rect. EK . GE,

= $(a + b)(a - b)$.

[i.e. $AB^2 - AC^2 = (AB + AC)(AB - AC)$.]

This fact can be enunciated thus:—

The differences of the squares on two unequal lines is equal to the rectangle contained by their sum and their difference.

EXAMPLES 70.

Illustrate (and then enunciate) theorems corresponding to the identities 1-7;

1. $a(a+b) = a^2 + ab$.

2. $(a+b)^2 = a(a+b) + b(a+b)$.

3. $(a+b)(x+y) = ax + bx + ay + by$.

4. $(x+2)(x+5) = x^2 + 7x + 10$.

5. $a(b-c) = ab - ac$.

6. $(2a)^2 = 4a^2$.

7. $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$.

8. Prove by means of (i) *algebraical* notation, (ii) *geometrical* notation, that, if AB be a line bisected at C, and divided unequally at D,

$$AD^2 + DB^2 = 2AC^2 + 2CD^2.$$

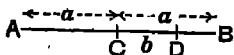


FIG. 249.

[(i) Let the lengths of AB, CD be $2a$, b , so that $AC = CB = a$.

$$\begin{aligned} \therefore AD &= a + b, \quad DB = a - b, \\ \therefore AD^2 + DB^2 &= (a+b)^2 + (a-b)^2, \\ &= a^2 + b^2 + 2ab + a^2 + b^2 - 2ab, \\ &= 2a^2 + 2b^2 = 2AC^2 + 2CD^2. \end{aligned}$$

Q.E.D.

$$\begin{aligned} \text{(ii)} \quad AD^2 &= AC^2 + CD^2 + 2AC \cdot CD, & (\text{AD div. int. at C}) \\ DB^2 &= CB^2 + CD^2 - 2CB \cdot CD, & (\text{DB div. ext. at C}) \\ &= AC^2 + CD^2 - 2AC \cdot CD, & (CB = AC, \text{ given}) \\ \therefore \text{by addition } AD^2 + DB^2 &= 2AC^2 + 2CD^2. \end{aligned}$$

Q.E.D.

There is no real difference between these proofs, but most pupils find the algebraical notation a little easier to use. Most of the riders following can be done in this way.]

9. XACB is a straight line, C being the mid point of AB; prove that $XC = \frac{1}{2}(XA + XB)$.

10. C is the mid point of a straight line AB, and O is any point in AC; prove that $OC = \frac{1}{2}(OB - OA)$.

11. If A, B, P, Q are points in a straight line, and if C, R bisect AB, PQ, prove that $2CR = AP + BQ$.

12. If A, B, C, D are four points in order on a straight line, show that $AB \cdot CD + AD \cdot BC = AC \cdot BD$.

13. If O, P, Q are points in a straight line and R bisects PQ, show that $OP \cdot OQ = OR^2 - PR^2$.

14. If O is the mid-point of AB and X any point in OB, prove that $AX^2 - XB^2 = 2AB \cdot OX$.

15. If AB is a line bisected at O and produced to X, prove that $AX^2 + XB^2 = 2AC^2 + 2CX^2$.

16. If a straight line is divided at two points, equally at one and unequally at the other, prove that the rectangle contained by the unequal segments is equal to the difference of the squares on (i) half the given line, and (ii) the line between the points of section.

17. If the line AB is bisected at O and produced to D, prove that $AD^2 = BD^2 + 4AO \cdot OD$.

18. If A, B, P, Q are points in a straight line and if O, R bisect AB, PQ, show that $2OR \cdot AB = AP \cdot AQ - BP \cdot BQ$.

19. AB is a straight line produced both ways to X and Y so that $AX = BY$; prove that $XY^2 + AB^2 = 2(AX^2 + XB^2)$.

20. If M and N are points in AB such that $AM^2 - MB^2 = AN^2 - NB^2$,

prove that M and N must coincide.

21. Four points P, Q, R, S are taken in order on a straight line, and on PS produced points Q', R' are taken such that $QS = SQ'$ and $RS = SR'$. Prove that the rectangle PQ · PQ' is less than the rectangle PR · PR'.

PROJECTION OF POINTS AND LINES ON A LINE.

If AB is a given line and X and Y are given points, and if XM, YN are drawn perpendicular to AB (or AB produced), then

(i) M is said to be the projection of the point X on the line AB.

(ii) MN is said to be the projection of the line XY on the line AB.

Notice that one point of the line projected may be on AB; thus the projection of MY on AB is MN; also the line projected may cross AB.

Ex. 1. In Fig. 250 what are the projections on AB of P, Q, R, T, QR, XQ, PR, MX?

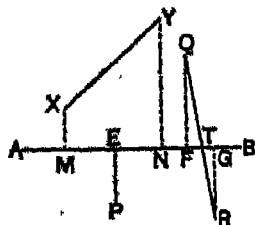


FIG. 250.

Ex. 2. In Fig. 251 what is the projection

(i) of AB on CD; (ii) of CD on AB?

Note that the projection on the ground of a straight tilted roof is the part of the ground that is dry when the rain is falling vertically (perpendicular to the ground).

Ex. 3. In the figure to Theorem 50, what are the projections of AB and AC on BC?

Ex. 4. In the first figure to Theorem 51, what are the projections of AC and AD on BC; what is the projection of AB on AD?

Ex. 5. Show that in an acute-angled Δ , the base is equal to the sum of the projections on the base of the other two sides.

A very useful truth in connection with Projection is:—

The projection of a line on another line inclined to it at an angle of 60° is half the original line.

Let AN be the projection of AC on AB where $\angle BAC = 60^\circ$.

To prove that $AN = \frac{1}{2}AC$.

Double the length of AN and so make AD = 2AN; join CD.

Now Δs ACN, DCN $\equiv$, (why?)

$\therefore \Delta$ ACD is isosceles,

and one of the base $\angle s$ (A) is 60° ,

$\therefore \Delta$ ACD is equilateral, (why?)

$\therefore AD = AC \therefore AN = \frac{1}{2}AC$.

Ex. 6. If the projection of one line on another is half the first line, prove that the acute angle between the lines is 60° .

Ex. 7. If the sides of a Δ are 1", 2", $\sqrt{3}$ ", prove that its angles are 30° , 60° , 90° .

[Pupils who have begun to learn Trigonometry will notice that the facts in the last three examples are the same as

$$\sin 30^\circ = \frac{1}{2}, \cos 60^\circ = \frac{1}{2}]$$

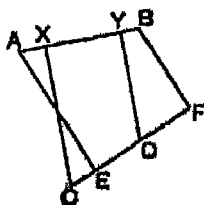


FIG. 251.

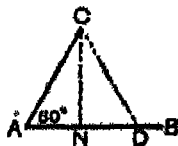
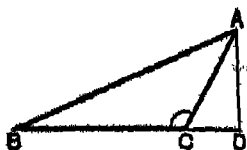


FIG. 252.

EXTENSION OF PYTHAGORAS—OBTUSE ANGLE.

THEOREM 50.

In an obtuse angled triangle, the square on the side subtending the obtuse angle is equal to the sum of the squares on the other two sides *together with twice the rectangle contained by one of those sides and the projection on it of the other.*



Given a $\triangle ABC$ with $\angle C$ obtuse, and a perp. AD from A to BC produced.

To prove that $AB^2 = BC^2 + CA^2 + 2BC \cdot CD$.*

Proof.

$$\begin{aligned}
 AB^2 &= AD^2 + BD^2, && \text{(Pythagoras)} \\
 &= AD^2 + CD^2 + BC^2 + 2BC \cdot CD, && \text{(BD div. int. at O)} \\
 &= AC^2 + BC^2 + 2BC \cdot CD && \text{for } AD^2 + CD^2 = AC^2 \text{ (Pythagoras)} \\
 &&& \text{Q.E.D.}
 \end{aligned}$$

Alternative Proof.

Let a, b, c be the sides of $\triangle ABC$;

let $AD = h, CD = p$.

$$\therefore AB^2 = h^2 + (a + p)^2,$$

$$\text{(Pythagoras)}$$

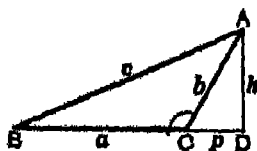
$$= h^2 + a^2 + 2ap + p^2.$$

$$\text{But } h^2 + p^2 = b^2, \text{ (Pythagoras)}$$

$$\therefore AB^2 = a^2 + b^2 + 2ap,$$

$$= BC^2 + CA^2 + 2BC \cdot CD.$$

Q.E.D.



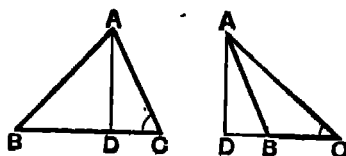
Cor. If in a triangle the square on one side is greater than the sum of the squares on the other two sides, the angle contained by these latter sides is obtuse.

* CD is here the projection of AC on BC (see enunciation).

EXTENSION OF PYTHAGORAS—ACUTE ANGLE.

THEOREM 51:

In any triangle, the square on the side subtending an acute angle is equal to the sum of the squares on the other two sides *diminished* by twice the rectangle contained by one of those sides and the projection on it of the other.



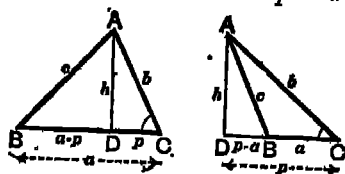
Given a ΔABC with $\angle C$ acute, and having $AD \perp BC$.
To prove that $AB^2 = BC^2 + CA^2 - 2CB \cdot CD$.

Proof.

$$\begin{aligned} AB^2 &= AD^2 + BD^2, && \text{(Pythagoras)} \\ &= AD^2 + CD^2 + BC^2 - 2BC \cdot CD, && \text{(BD div. ext. at C)} \\ &= AC^2 + BC^2 - 2BC \cdot CD. && \text{(Pythagoras)} \end{aligned}$$

Q.E.D.

Alternative Proof. Let a, b, c be the sides of ΔABC ;
let $AD = h, CD = p. \therefore BD = a - p$ in Fig. i
or $p - a$ in Fig. ii.



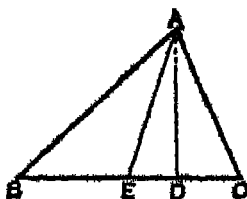
$$\begin{aligned} \text{Then } AB^2 &= h^2 + (a - p)^2, && \text{(Pythagoras)} \\ &= h^2 + a^2 - 2ap + p^2. \\ &\quad \text{But } h^2 + p^2 = b^2, && \text{(Pythagoras)} \\ \therefore AB^2 &= a^2 + b^2 - 2ap, \\ &= BC^2 + CA^2 - 2CB \cdot CD. && \text{Q.E.D.} \end{aligned}$$

Cor. If in a triangle the square on one side is less than the sum of the squares on the other two sides, the angle contained by these latter sides is acute.

APOLLONIUS' THEOREM.

THEOREM 52.

In any triangle, the sum of the squares on any two sides is equal to twice the square on half the third side, together with twice the square on the median which bisects the third side.



Given a $\triangle ABC$ in which AE is the median from A to BC .

To prove that $AB^2 + AC^2 = 2BE^2 + 2AE^2$.

Constr. Draw $AD \perp BC$ (or BC produced).

Proof. Except in the special case of $\triangle ABC$ being * isosceles, the angles at E are not equal.

Let $\angle AEB$ be the obtuse $\angle$.

Then in $\triangle ABE$, $\angle AEB$ is obtuse,

$$\therefore AB^2 = BE^2 + AE^2 + 2BE \cdot ED; \quad (i)$$

also in $\triangle AEC$, $\angle AEC$ is acute,

$$\therefore AC^2 = EC^2 + AE^2 - 2CE \cdot ED, \quad (ii)$$

$$\text{which} = BE^2 + AE^2 - 2BE \cdot ED,$$

since $EC = BE$

(given)

$$\text{Add } \therefore AB^2 + AC^2 = 2BE^2 + 2AE^2.$$

Q.E.D.

Cor. The difference of the squares on any two sides of a triangle is equal to twice the rectangle contained by the base and the projection on the base of the median bisecting the base.

For subtracting line (ii) from line (i) above, we get

$$AB^2 - AC^2 = 4BE \cdot ED = 2BC \cdot ED.$$

* What does this Theorem become in the case of an isosceles $\triangle$?

EXAMPLES 71.

NUMERICAL.

1. Find whether a triangle is obtuse-angled, right-angled, or acute-angled if its sides are—
 (i) 5, 6, 8; (ii) 7, 8, 10; (iii) 10, 11, 15; (iv) 8, 15, 17;
 (v) 10, 11, 14.
2. In a $\triangle ABC$, $AB = 3$, $AC = 2$, $BC = 4$ and $AX \perp BC$; calculate BX and CX .
3. In $\triangle ABC$, AX is an altitude, and $AB = 7$, $AC = 5$, $BC = 3$; calculate BX and CX .
4. In $\triangle ABC$, $AB = 13$, $BC = 15$, $CA = 14$. If BD is an altitude, calculate AD , CD , BD . Hence find the area of the $\triangle$.
5. Find the area of a $\triangle$ whose sides are 7, 10, 13.
6. In $\triangle ABC$, $AC = 13$, $AB = 8$, $BC = 15$; prove $\angle B = 60^\circ$.
7. In $\triangle ABC$, $AB = 7$, $BC = 5$, $CA = 3$; prove $\angle C = 120^\circ$.
8. In $\triangle ABC$, $AC = 7''$, $BC = 3''$, $\angle C = 60^\circ$; find AB .
9. In $\triangle ABC$, two sides are 8 cm. and 7 cm. and the included $\angle$ is 120° ; find the third side.
10. Find the lengths of the medians of a $\triangle$ (correct to $\cdot 1''$) if the sides are (i) $4''$, $6''$, $8''$; (ii) $6''$, $9''$, $10''$.
11. Calculate the length of the median to the longest side in the case of a $\triangle$ whose sides are $8''$, $15''$, $17''$. What does the answer suggest about the $\triangle$?
12. Two sides of a $\triangle$ are $5''$ and $7''$ and the median to the base is $2''$; find the length of the base (correct to $\cdot 01''$).
13. A parallelogram has its sides 5 cm. and 7 cm. long and one diagonal is 9 cm. long; calculate the length of the other.
14. The diagonals of a parallelogram are $22''$ and $4''$ and one side is $10''$; find the length of the adjacent side.
15. A and B are two consecutive milestones on a straight level road and C is a balloon which is overhead at D, some point in the road. If $BC = 1840$ yds. and CA is 400 yds., find whether D lies between A and B or not.

THEORETICAL.

16. ABC is a $\triangle$ and AX , BY are altitudes; prove that
 $BC \cdot CX = AC \cdot CY$.
17. ABC is a $\triangle$ in which $AB = AC$ and $EX \perp AC$; prove that
 $BC^2 = 2CA \cdot CX$.

18. $\triangle ABC$ is a $\triangle$ in which $\angle B = 60^\circ$; prove that

$$AC^2 = AB^2 + BC^2 - AB \cdot BC.$$

19. If D be any point in the base BC of an isosceles $\triangle ABC$, prove that $AB^2 - AD^2 = BD \cdot DC$.

20. $\triangle ABC$ is a $\triangle$ in which $\angle A = 90^\circ$, and $AD \perp BC$; prove by Theorem 51 that $AC^2 = BC \cdot CD$.

21. Prove that, if P is any point in the plane of a rectangle $ABCD$,
 $PA^2 + PC^2 = PB^2 + PD^2$.

22. Prove Ex. 21 when P is outside the plane of the rectangle.

23. Show that the sum of the squares on the sides of a parallelogram is equal to the sum of the squares on its diagonals.

24. Find the locus of a point P which moves so that $PA^2 + PB^2$ is constant, A and B being two fixed points.

25. $\triangle ABC$ is a $\triangle$ and the median $AD =$ the side AB ; prove that
 $AC^2 = AD^2 + 2BD^2$.

26. $\triangle ABC$ is a $\triangle$ whose base BC is trisected at X and Y ; prove that
 $AB^2 + AC^2 = AX^2 + AY^2 + 4XY^2$.

27. The sum of the squares on the four sides of a quadrilateral is equal to the sum of the squares on its diagonals $+ 4XY^2$, if X and Y bisect the diagonals.

28. $\triangle ABC$ is a $\triangle$ in which $AB = AC$ and DE is drawn $\parallel BC$ to meet AB in D and AC in E ; prove that $CD^2 - CE^2 = BC \cdot DE$.

29. $\triangle XYZ$ is a $\triangle$ with $XD \perp YZ$ and E the mid-point of YZ . If ED is $\frac{1}{2}YZ$, prove that the difference of the squares on XY and $XZ = \frac{1}{4}YZ^2$.

30. If $\triangle APB$ and $\triangle AQB$ are $\triangle$ s on the same base AB and on opposite sides of it, and if $AP^2 - PB^2 = AQ^2 - QB^2$, prove that $PQ \perp AB$. [Draw perpendiculars PN, QM to AB and show that N and M coincide.]

31. A point P moves so that $PA^2 + PB^2 + PC^2 + PD^2$ is constant, A, B, C, D being fixed points. Prove by successive applications of Apollonius that the locus of P is a fixed circle, assuming all the points to lie in a plane.

32. Use Apollonius to prove that if two medians of a $\triangle$ are equal, the $\triangle$ is isosceles.

33. On the side BC of a $\triangle ABC$ a point D is taken such that $BD = 2DC$; prove that $AB^2 + 2AC^2 = 3AD^2 + 6DC^2$.

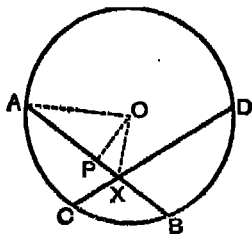
34. OA, OB, OC are the three straight lines mutually at right angles where the three walls of a room meet in O ; prove that the $\triangle ABC$ must be acute-angled.

35. If a, b, c are the sides of a $\triangle$ and x is the median bisecting c , prove that $x^2 = \frac{1}{2}b^2 + \frac{1}{2}a^2 - \frac{1}{4}c^2$.

* SEGMENTS OF CHORDS.

THEOREM 53.

If two chords of a circle intersect at a point inside the circle, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.



Given two chords of a circle, centre O , which intersect at X within the circle.

To prove that $AX \cdot XB = CX \cdot XD$.

Constr. Draw $OP \perp AB$ from the centre; join OA , OX .

Proof. Since $OP \perp AB$, $AP = PB$.
 Then $AX \cdot XB = (AP + PX)(PB - PX)$,
 $= (AP + PX)(AP - PX)$, for $AP = PB$,
 $= AP^2 - PX^2$
 $= (AO^2 - OP^2) - (OX^2 - OP^2)$, (*Pythagoras*)
 $= AO^2 - OX^2$,
 $= r^2 - OX^2$, where r is the radius.

Similarly $OX \cdot XD = r^2 - OX^2$,

$\therefore AX \cdot XB = CX \cdot XD$.

Q.E.D.

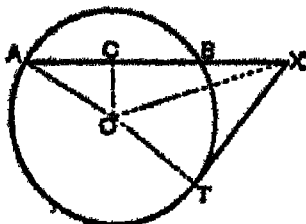
Cor. If, through a point X inside a circle, centre O and radius r , a chord is drawn, the rectangle contained by the segments of the chord is $r^2 - OX^2$.

* For Alternative Proof of Theorem 53 see page 248.

* SECANT AND TANGENT.

THEOREM 54.

If from a point outside a circle a secant and a tangent to the circle are drawn, the rectangle contained by the whole secant and the segment of it outside the circle is equal to the square on the tangent.



Given a circle, centre O, and a point X outside it, and from X a secant XBA and a tangent XT to the circle.

To prove that $AX \cdot XB = XT^2$.

Constr. Draw $OC \perp AB$ from the centre; join OA, OX, OT.

Proof.

Since $OC \perp AB$, $AC = CB$.

$$\begin{aligned}
 \text{Then } AX \cdot XB &= (XC + CA)(XC - CB), \\
 &= (XC + CA)(XC - CA), \text{ for } CA = CB \\
 &= XO^2 - CA^2, \\
 &= (XO^2 - OC^2) - (OA^2 - OC^2), \text{ (Pythagoras)} \\
 &= XO^2 - OA^2, \\
 &= XO^2 - OT^2, \quad (OA = OT, \text{ radii}) \\
 &= XT^2, \quad (\text{Pythagoras; } \text{tgt. } \perp \text{ rad.})
 \end{aligned}$$

Q.E.D.

Cor. 1. If from a point X outside a circle, centre O and radius r, a secant is drawn to the circle, the rectangle contained by the whole secant and the segment of it outside the circle $= OX^2 - r^2$.

* For Alternative Proof of Theorem 54 see page 249.

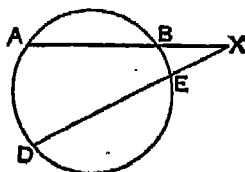


FIG. 254A.

Cor. 2. If from a point outside a circle two secants are drawn to the circle, the rectangles contained by the whole secants and their segments outside the circle are equal.

For in Fig. 254A, $\text{rect. } AX.XB = \text{rect. } DX.XE$, since each rectangle $= OX^2 - r^2$.

The converses of Theorems 53 and 54 are true and can be proved at once by the ordinary Reductio ad Absurdum Method. There is a modification of this method which can also be used very conveniently. It is shown below.

Converse of Theorem 53. If two lines AB, CD intersect at X so that $\text{rect. } AX.XB = \text{rect. } CX.XD$, then the points A, B, C, D are concyclic.

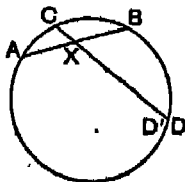


FIG. 255.

For draw a circle through A, B, C and suppose that it meets CX produced in D'.

Then $\because$ AB and CD' are two chords of a circle,

$$\therefore AX.XB = CX.XD'.$$

$$\text{But } AX.XB = CX.XD,$$

$$\therefore CX.XD' = CX.XD,$$

$$\therefore XD' = XD,$$

$\therefore$ D' coincides with D. But the circle ABC goes through D', $\therefore$ it goes through D, i.e. A, B, C, D are concyclic.

Q.E.D.

Converse of Theorem 54. If from a point outside a circle a secant is drawn and also a line meeting the circle, so that the rectangle contained by the whole secant and the segment of it outside the circle equals the square on the line meeting the circle, then this latter line is a tangent to the circle.

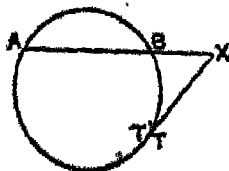


FIG. 155.

For if a secant XAB is drawn from X , outside a circle, and also a line XT to meet the circle in T , so that $XA \cdot XB = XT^2$, we have to prove that XT is a tangent to the circle.

Let XT meet the circle again in T' .

Then $\because XBA$ and XTT' are two secants,

$$\therefore XA \cdot XB = XT \cdot XT',$$

$$\text{But } XA \cdot XB = XT^2,$$

(given)

$$\therefore XT \cdot XT' = XT^2 \therefore XT' = XT \therefore T' \text{ and } T \text{ coincide,}$$

$\therefore XT$ meets the circle in two coincident points, i.e. XT is a tangent to the circle at T .

Q.E.D.

SEGMENTS OF CHORDS AND SECANTS.

EXAMPLES 72.

NUMERICAL.

1. AB and CD are two chords of a circle crossing at X ; if $AX = 3$ cm., $XB = 5.5$ cm., $OX = 3.3$ cm., calculate XD .

2. From an external point P two secants PAB and PMN are drawn to a circle; if $PA = 1.8''$, $AB = 1.7''$, and $PM = 1.4''$, calculate MN .

3. AB is a chord of a circle of radius 12 cm. passing through a point O inside a circle and $AO = 5$ cm., $OB = 16$ cm. How far is O from the centre?

4. Answer Ex. 3 if O is outside the circle (correct to .1 cm.).

5. Find the length of the tangents drawn to a circle from a point O outside a circle if a secant OAB is drawn to the circle such that $OA = 9$ cm., $AB = 7$ cm.

6. Through a fixed point P inside a circle of radius $2''$ a chord CPD is drawn such that CP is twice PD ; find the length of CD if P is $1''$ from the centre of the circle. (Ans. to $1''$.)

7. O is a point outside a circle at a distance of 9 cm. from the centre, and from O a line OAB is drawn to meet the circle in A and B so that $OA = AB = 4$ cm. Find (i) the radius of the circle; and (ii) the length of the tangent from O to the circle.

8. The chord of an arc is $18''$ and the height of the arc is $6''$. Find the radius of the circle.

9. A cart-wheel just grazes a vertical step four inches high which is distant $14''$ away from the point where the wheel touches the ground. Find the diameter of the wheel.

10. ABC is a Δ in which $AB = 7$ cm., $BC = 6$ cm., $CA = 9$ cm. If D is a point in AC such that $AD = 5$ cm., prove that the circle ABD touches BC at E .

11. ABC is a Δ right-angled at C and the circle on AC as diameter cuts AB in D ; if $BC = 3$ cm. and $BD = 2$ cm., find AD and AC .

12. Two chords of a circle cut at right angles internally; the segments of one chord are $6''$ and $3''$ and one segment of the other chord is $9''$; find the radius of the circle.

THEORETICAL.

13. Two circles meet in A and B , and C is any point in the chord AB . Through C are drawn chords PQ , RS , one in each circle; show that $PC \cdot CQ = RC \cdot CS$.

14. In the common chord AB of two circles when produced a point X is taken, and XT , XV are tangents from X , one to each circle; prove that $XT = XV$.

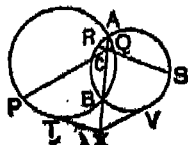


FIG. 257.

15. AD , BE are two altitudes of ΔABC meeting in O ; prove that $AO \cdot OD = BO \cdot OE$.

16. Prove in the figure to Ex. 15 that $CA \cdot CE = CD \cdot CB$.

17. AOE is a diameter of a circle and from a point F on the circumference a perpendicular FD to AE is drawn; prove that $FD^2 = AD \cdot DE$.

18. Prove that a common tangent to two intersecting circles is bisected by the common chord produced.

19. Prove that, if AB , CD are two lines which meet when produced at X so that $AX \cdot XB = CX \cdot XD$, a circle can be drawn through A , B , C , D .

30. X is a point on the common chord produced of two circles, and from X are drawn two secants XAB , XCD , one in each circle; prove that A , B , C , D are concyclic.

31. Two chords AB , CD are drawn in a circle and a larger concentric circle cuts the chords, when produced, in X , Y respectively; prove that $AX \cdot XB = OY \cdot YD$.

32. $ABCD$ is a square and E is any point on BC produced; if P is taken on AE so that $AP \cdot AE = AB^2$, find the locus of P as E moves along BC produced.

33. ABC is a Δ in which $AB = AC$, and a circle is drawn to pass through B and to touch AC at its middle point; if the circle cuts AB in X , prove that $BX = 3AX$.

34. AB , AC are two chords of a circle; a line parallel to the tangent at A cuts AB in X and AC in Y . Show that $AB \cdot AX = AC \cdot AY$.

35. Find the locus of the points of contact of tangents drawn from the extremity A of a given line ABC to circles drawn through B and C .

36. If P be a point from which the tangents to two intersecting circles are equal, prove that P must lie on the common chord AB produced. (Show that PA produced must meet the two circles in the same point.)

37. BAC is a Δ right angled at A , and $AD \perp BC$; if $BA = 3AC$, prove that $BD = 9DC$.

38. A circle touches the arms AB , AC of an angle BAC at X and Y . O is any point on XY between X and Y and any line through O meets AB in X' and AC in Y' . Prove that $XO \cdot OY < X'O \cdot OY'$.

39. A tangent CA to a circle at C is bisected at B and through B a secant BDE is drawn to the circle; prove that the centre of the circle through A , D , E is on the perpendicular to AC through A .

40. Three circles are drawn so that each intersects the other two; prove that their common chords are concurrent. (If two common chords meet in O , prove that the line joining O to one end of the third common chord must meet the two circles concerned in the same point.)

41. If P is any point on the circle circumscribing a rectangle $ABCD$ and PX , PY , PZ , PW are perpendicular to AB , BC , CD , DA , prove that $PX \cdot PZ = PY \cdot PW$.

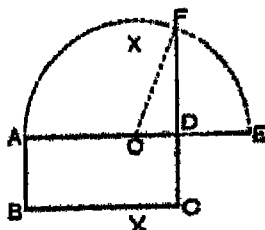
42. A , B , C , D are four points in order on a straight line. Show how to find a point O on the line between B and C such that $OA \cdot OC = OB \cdot OD$.

PROBLEM 14.

To construct a square equal in area to a given rectangle.

Given a rectangle ABCD.

To draw a square equal in area to ABCD.



Constr. Produce AD to E, making DE = DC.

Bisect AE at O and draw a semicircle on AE as diameter.

Produce OD to meet circle in F, and draw a sq. on DF.

Then shall this square be the sq. required.

Proof. If r be the radius of the circle,

$$\text{rect. } ABCD = \text{rect. } AD \cdot DE,$$

$$= (r + OD)(r - OD),$$

$$= r^2 - OD^2,$$

$$= DF^2.$$

(Pythagoras)

$\therefore$ the sq. on DF is the sq. required.

Q.E.D.

[An alternative proof is found in Ex. 17 on page 219.

A third proof follows at once from Theorem 61.]

N.B.—It is now possible to construct a square equal in area to a given polygon, for—

(i) the polygon can be turned into a triangle;

(cutting off one corner at a time; see p. 122)

(ii) the triangle into a rectangle;

(taking half the base or half the height; see p. 123)

(iii) the rectangle into a square.

(as above)

EXAMPLES 74.

PRACTICAL.

(As a rule only one side of the required square is drawn; it is then measured.)

Construct a square equal in area to—

1. A rectangle measuring $2\frac{1}{2}$ " by 7 ".
2. A rectangle measuring 5 cm. by $3\frac{1}{2}$ cm.
3. An equilateral triangle of side 2 ".
4. A triangle whose sides are $1\frac{1}{4}$ ", $1\frac{1}{2}$ ", $1\frac{3}{4}$ ".
5. A triangle whose sides are 5 cm., 6 cm., 7 cm.
6. A kite whose sides are $1\frac{1}{2}$ ", $1\frac{1}{2}$ ", 2 ", 2 ", the angle between the first two sides being 110° .
7. A regular pentagon inscribed in a circle of radius $1\frac{1}{4}$ ".
8. A regular hexagon of side 4 cm.
9. A quadrilateral ABCD in which $AB = 2\frac{1}{2}$ ", $BC = 1\frac{1}{2}$ ", $\angle ABD = 60^\circ$, $\angle CBD = 75^\circ$, $\angle DCB = 75^\circ$.
10. Use the construction given in Problem 14 to solve the equations $x + y = 12$, $xy = 24$.

SECTION VI.

RATIO.

The meaning of the word ratio is familiar in Arithmetic. If two straight lines are taken, whose lengths are 6 ins. and 9 ins., the first is $\frac{2}{3}$ of the second, and we may say that the ratio of their lengths is $\frac{\text{first length}}{\text{second length}} = \frac{2}{3} = \frac{2}{3} = 2:3$. Notice that we can find the ratio of inches to inches, but not of inches to sq. yds. or lbs. The units must be *of the same kind*. The ratio of things of the same kind is merely a comparison of their magnitudes. Note that the ratio of 6s. to 5d. is 60:5 or 12:1, since they may both be expressed in pence.

If we take four quantities a, b, c, d such that the ratio of a to b equals the ratio of c to d , i.e. $\frac{a}{b} = \frac{c}{d}$, the quantities are said to be in proportion, and the fourth (d) is said to be the fourth proportional to a, b, c .

If we take three quantities a, b, c such that a, b, b, c are in proportion (i.e. $\frac{a}{b} = \frac{b}{c}$), the three quantities are said to be in continued proportion, and c is said to be the third proportional to a and b . Also b is said to be the mean proportional to a and c .

The word "*continued*" is sometimes omitted since three quantities cannot form a proportion unless one is repeated. Why is the second one the one that is repeated?

EXAMPLES 75.

1. Find the values of the ratios: (i) 1x. to 2d.; (ii) 3 hrs. to 1 day; (iii) 2 sq. ft. to 1 sq. yd.; (iv) $7\frac{1}{2}$ d. to £1; (v) 1 in. to 4 cm. if 1 in. = $2\frac{1}{2}$ cm.; (vi) 75° to 2 rt. $\angle$ s; (vii) 15 cm. to 2 m.

2. If $\frac{a}{b} = \frac{c}{d}$, prove that— (i) $\frac{a}{c} = \frac{b}{d}$; (ii) $ad = bc$;

(iii) $\frac{b}{a} = \frac{d}{c}$; (iv) $\frac{a+b}{b} = \frac{c+d}{d}$;

(v) $\frac{a}{a+b} = \frac{c}{c+d}$; (vi) $\frac{a-b}{a+b} = \frac{c-d}{c+d}$.

3. Are the following quantities in proportion?

- (i) 5", 1', 15", 3'; (ii) 1 sq. in., 1 sq. foot, 5d., £3;
(iii) 4, 8, 9, 12; (iv) 4d., 6d., 9 inches, 12 inches.

4. Find x given that—

- (i) $3:4 = 5:x$; (ii) $5:8 = x:4$; (iii) $x':3'' = 4:8'$;
(iv) $3:x = x:12$; (v) $4:x = x^2:2$.

5. Are 3", 1", 1 yd., 1 ft. in continued proportion?

6. Find the fourth proportional to—

- (i) 5, 8, 15; (ii) 4", 3', 4'; (iii) ab , ac , bc .

7. Find the third proportional to—

- (i) 5, 24; (ii) 12, 8; (iii) a^2 , ab .

8. Find the mean proportional to—

- (i) 5, 24; (ii) 4", 3'; (iii) a^2b , b^2 .

9. The sides of a triangle are in the ratio 2:3:4 and the longest side is x'' ; find the other sides.

DIVISION OF A STRAIGHT LINE IN A GIVEN RATIO.

If AB is a line divided internally at O its segments are AO and OB and it is said to be divided internally in the ratio $AO:OB$, or $\frac{AO}{OB}$.

(See Fig. I.)



FIG. I.



FIG. II.
FIG. 262.

If AB is a line divided externally at O' , its segments are AO' and $O'B$, and the line is said to be divided externally in the ratio $AO':O'B$, or $\frac{AO'}{O'B}$. (See Fig. II.)

Notice that in the ratio in which a line is divided, the line itself does not occur, but only its segments; also notice that in the first figure BA is divided internally at C in the ratio BC:CA, and in the second figure BC' is divided externally at A in the ratio BA:AC'.

EXAMPLES 77.

1. In Fig. 262 (ii) what is the ratio in which AC' is divided at B? and in which C'B is divided at A?

2. In Fig. 262 (i) what is the ratio in which CB is divided at A? and in which AC is divided at B?

3. Draw a line AB which is about 1 in. long, and guess the positions of points in which it is divided—

- (i) internally in the ratio 1:3 (C);
- (ii) internally in the ratio 3:2 (D);
- (iii) externally in the ratio 2:1 (E);
- (iv) externally in the ratio 2:3 (F).

4. Give a rough guess of the ratio in which in Fig. 262 (i) AB is divided at C, and in Fig. 262 (ii) AB is divided externally at C'.

5. Is it possible for a given line AB to be divided internally in a given ratio say 2:3 (or $x:y$) in more than one point?

[This is equivalent to dividing a line into 5 equal parts and taking AC to be 2 parts and CB to be 3 parts; thus $AC = \frac{2}{5}$ of the whole line and $CB = \frac{3}{5}$ of the whole line. Can a second suitable point C' be found internally?]

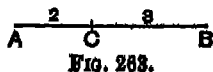


FIG. 263.

6. Is it possible for a given line AB to be divided externally in a given ratio, say 5:2?

[Will the point required be beyond B or

beyond A? If D be the point, $\frac{AD}{DB} = \frac{5}{2}$;

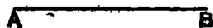


FIG. 264.

$\therefore$ AD will be 5 units, say $5x$, and DB will be 2 units, viz. $2x$. Hence D will be beyond B and AB will be $3x \therefore x = \frac{1}{3}AB$.

$\therefore$ D can only have one position, beyond B, and $AD = \frac{5}{3}AB$, and $BD = \frac{2}{3}AB$.]

7. If a line AB be divided internally at X in the ratio 4:7, will X be nearer to A or to B? What fraction of the line is (i) AX, (ii) XB?

8. If a line AB be divided externally at Y in the ratio $2:7$, will Y be beyond A or beyond B ? What fraction of the line is (i) AY , (ii) YB ?

9. A line AB $2''$ long is divided internally at C and externally at D in the ratio $3:7$; calculate AC , CB , AD , DB . Draw a rough figure.

10. A line a'' long is divided internally in the ratio $p:q$; find the lengths of its segments.

11. A line b'' long is divided externally in the ratio $12:5$; find the lengths of its segments.

12. A line a'' long is divided internally at P and externally at Q in the ratio $m:n$, where $m > n$; find the length of the line PQ .

PRELIMINARY THEOREM.

The areas of two triangles having equal altitudes are to one another as their bases.

Given two Δ s ABO , DEF , having equal altitudes AX , DY .

To prove that $\frac{\Delta ABO}{\Delta DEF} = \frac{BO}{EF}$.

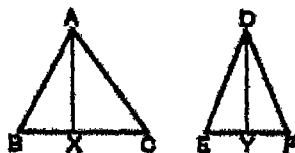


FIG. 355.

$$\begin{aligned} \text{Proof. } \frac{\Delta ABO}{\Delta DEF} &= \frac{\frac{1}{2}AX \cdot BO}{\frac{1}{2}DY \cdot EF} \quad (\text{Area of } \Delta = \frac{1}{2} \text{ base} \times \text{alt.}) \\ &= \frac{BO}{EF}. \quad (\text{since } AX = DY) \end{aligned}$$

Q.E.D.

Cor. The areas of two triangles having equal bases are to one another as their altitudes.

Note on Incommensurables. Definition. Two magnitudes are said to be incommensurable if their ratio cannot be expressed exactly as the ratio of two whole numbers.

Thus a diagonal of a square and its side are incommensurable, for their ratio is $\sqrt{2} : 1$ (by Pythagoras), and as $\sqrt{2}$ cannot be expressed exactly as a decimal we cannot express $\sqrt{2} : 1$ exactly as the ratio of two numbers. [We could, however, take $\sqrt{2}$ as 1.414 very nearly, and so the ratio would be 1.414 : 1 i.e. 1414 : 1000 approx.]

It is not easy to prove the Theorems dealing with ratio to be true for incommensurables. We can, however, prove them true first of all for commensurables, and we can then find commensurable quantities which are *almost* equal to the given incommensurables (such as 1.414" and $\sqrt{2}$ "), and so we learn the truth of these Theorems for quantities that are almost exactly, but not quite, equal to the given incommensurable quantities.

EXAMPLES 78.

1. Prove in full that triangles on equal bases are to one another as their altitudes.

2. D is a point in the base BC of $\triangle ABC$; show that

$$\triangle ABD : \triangle ACD = BD : CD.$$

3. X is a point on the side AB of $\triangle ABC$ such that $AX = \frac{1}{3}AB$. How many times is $\triangle ABC$ greater than $\triangle AXC$?

4. In $\triangle ABC$ the point P divides BC so that $BP = \frac{1}{3}BC$ and Q divides CA so that $CQ = \frac{1}{3}CA$. Prove that

$$\triangle AQP = \frac{1}{3} \triangle ABC.$$

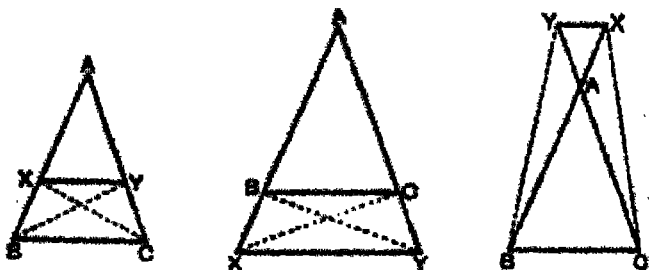
5. In $\triangle ABC$, XY is drawn $\parallel AB$ to cut AC in X and BC in Y. Prove that, if AY, BX are joined—

$$(i) \frac{\triangle CXY}{\triangle AXY} = \frac{CX}{XA}; \quad (ii) \frac{\triangle CXY}{\triangle BXY} = \frac{CY}{YB}; \quad (iii) \frac{CX}{XA} = \frac{CY}{YB}.$$

6. In a $\triangle ABC$, D is a point on BC such that $BD = 4''$, $BC = 5.2''$; find the values of the ratios (i) $\frac{\triangle ABD}{\triangle ADC}$, (ii) $\frac{\triangle ABC}{\triangle ABD}$.

THEOREM 55.

A straight line drawn parallel to one side of a triangle divides the other sides proportionally.



Given a $\triangle ABC$ with XY drawn $\parallel BC$ and cutting AB in X and AC in Y .

To prove that $\frac{AX}{XB} = \frac{AY}{YC}$.

Constr. Join BY , CX .

Proof. Then $\frac{AX}{XB} = \frac{\triangle AXY}{\triangle BXY}$ ($\triangle$ s having same alt.)

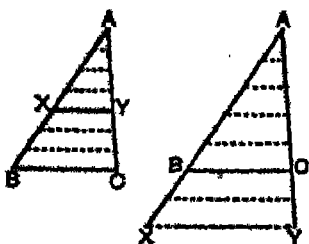
and $\frac{AY}{YC} = \frac{\triangle AXY}{\triangle CXY}$ (.)

But $\triangle BXY = \triangle CXY$, (same base XY and bet. same $\parallel$ s)

$$\therefore \frac{AX}{XB} = \frac{AY}{YC}$$

Alternative Proof. Suppose that $\frac{AX}{XB} = \frac{m}{n}$, where m and n are whole numbers.

Divide AX into m equal parts; then XB can be divided into n equal parts of the same size.



Through the points of division draw lines $\parallel BC$ to meet AC .

Then the $\parallel$ s cut off equal intercepts from AC, and AY contains m of them, and YC n of them.

$$\therefore \frac{AY}{YC} = \frac{m}{n},$$

$$\therefore \frac{AX}{XB} = \frac{AY}{YC}.$$

Cor. If $XY \parallel BC$, $\frac{AX}{AB} = \frac{AY}{AC}$ and $\frac{AB}{BX} = \frac{AC}{CY}$.

These can be proved in a similar way or they can be deduced from the result of the Theorem.

THEOREM 56.

If two sides of a triangle are divided in the same ratio, the straight line joining the points of section is parallel to the third side.

* Given a ΔABC in which X and Y divide AB and AC so that

$$\frac{AX}{XB} = \frac{AY}{YC}.$$

To prove that $XY \parallel BC$.

Constr. Join BY, CX.

Proof. Then $\frac{AX}{XB} = \frac{\Delta AXY}{\Delta BXY}$, (Δ s having same alt.)

$$* \quad \text{and } \frac{AY}{YC} = \frac{\Delta AXY}{\Delta CXY}.$$

$$\text{But } \frac{AX}{XB} = \frac{AY}{YC}, \quad (\text{given})$$

$$\therefore \frac{\Delta AXY}{\Delta BXY} = \frac{\Delta AXY}{\Delta CXY},$$

$$\therefore \Delta BXY = \Delta CXY.$$

$$\therefore XY \parallel BC.$$

Q.E.D.

An ALTERNATIVE PROOF of Theorem 56, which does not use the Theorem on page 226, will be found at the bottom of page 252.

* Use the first three Figures on page 228.

EXAMPLES 79.

NUMERICAL.

1. In the first figure to Th. 55:—

- (i) If $AX = 1.4''$, $XB = .4''$, $AY = 2.1''$, find YC ;
 (ii) If $AB = 2.4''$, $AC = 3.6''$, $AX = 1.4''$, find AY ;
 (iii) If $AX = 1.8''$, $AB = 2''$, $YC = .5''$, find AC .

2. In Th. 55 (Fig. 3). If $AY = 2.4''$, $AC = 3''$, $BX = 4.8''$, find AX .

3. In Th. 55 (Fig. 1), if $\frac{AX}{AB} = \frac{1}{4}$, find $\frac{AB}{BX} \cdot \frac{AY}{YC} \cdot \frac{AC}{CY} \cdot \frac{AC}{AY}$.

4. In Th. 55 (Fig. 3), if $\frac{AY}{YC} = \frac{1}{2}$, find $\frac{AY}{AC} \cdot \frac{AX}{XB} \cdot \frac{AB}{BX}$.

THEORETICAL.

5. AOB is a straight line and semicircles are drawn on AO , OB as diameters; if a line AHF through A meets these semicircles in E , F , prove that $AE : EF = AO : OB$.

6. $ABCD$ is a trapezium with AB parallel to DC , and a line parallel to AB cuts AD in X and BC in Y ; prove that $AX : XD = BY : YC$.

7. ABC , DCB are triangles on opposite sides of the same base BC ; X is a point in BC , and XY , XZ are drawn parallel to BA , CD to meet AC , AD in Y , Z ; prove that YZ is parallel to AD .

8. $ABCD$ is a parallelogram; E and F bisect AB and CD ; prove that AC is bisected by DE and BF .

9. In $\triangle ABC$ points D , E are taken on AB , AC such that $AD = \frac{1}{3}AB$ and $CE = \frac{1}{3}CA$; prove that a line from C drawn parallel to DE bisects AB .

10. O is a point inside an angle ABC ; show how to draw a line through O meeting AB in X and BC in Y such that $OX : OY = 3 : 1$.

11. ABC is a triangle; X is a point on AB and Y on AC such that $AX = 2XB$ and $OY = 2YA$. If CX , BY meet in O , prove that (i) $BO = \frac{1}{2}OY$, and (ii) $CO = 6OX$. [To prove (i) draw $YZ \parallel CX$.]

12. X and Y are points on the sides AB , AC of $\triangle ABC$ such that $BX : XA = 3 : 1$ and $AY : YC = 3 : 2$. If BY , CX meet in O , prove that (i) $CO = OX$ and (ii) $BO = 5OY$.

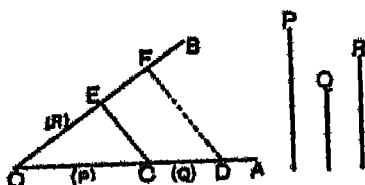
13. XYZ is a triangle; M is a point on XY and N on XZ and MZ , NY meet in O . If $YM = 3MX$ and $YQ = 6QN$, prove that $XN = 3NZ$. Find also the ratio $MQ : QZ$.

PROBLEM 15.

To construct a fourth proportional to three given straight lines.

Given three straight lines P, Q, R.

To make a fourth proportional to P, Q, R.



Constr. Make an angle AOB of any size;
 along OA mark off OC equal to P, and CD equal to Q;
 along OB mark off OE equal to R.
 Join CE and draw DF $\parallel$ CE to meet OB in F.
 Then EF shall be the fourth prop^l required.

Proof. Since CE $\parallel$ DF $\therefore \frac{OC}{OD} = \frac{OE}{EF}$,

$$\text{i.e. } \frac{P}{P+Q} = \frac{R}{R+EF}$$

$\therefore$ EF is a fourth prop^l to P, Q, R.

Cor. To construct a third proportional to two given lines (P, Q).

In the above we put R equal to Q, and so obtain

$$\frac{P}{Q} = \frac{Q}{EF}$$

$\therefore$ EF is a third prop^l to P, Q.

Q.E.D.

PROBLEM 16.

To divide a given straight line, internally or externally, in a given ratio ($k:l$).

Given a straight line AB .

To divide it (i) internally, (ii) externally in the ratio $k:l$.

(i) Internally. Draw through A any line AO making an angle with AB ;

along AO mark off $AX = k$ units, and further $XY = l$ units.

$\therefore AY = k + l$ units.

Join BY , and draw $XO \parallel YB$, to meet AB in O .

Then O is the required point.

Proof. Since $XO \parallel YB$, $\frac{AO}{OB} = \frac{AX}{XY} = \frac{k}{l}$.

Q.E.F.

(ii) Externally. (Assume $k > l$.) Draw AO through A as before;

along AO mark off $AX = k$ units, and then XZ backwards $= l$ units.

$\therefore AZ = k - l$ units.

Join ZB , and draw $XO \parallel ZB$, to meet AB produced in O .

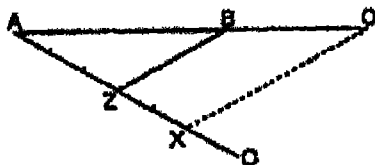
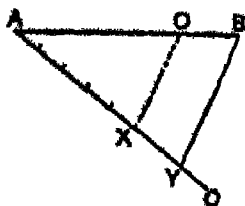
Then O is the required point.

Proof. Since $XO \parallel ZB$, $\frac{AO}{OB} = \frac{AX}{XZ} = \frac{k}{l}$.

Notes. (a) The figures are drawn to suit the cases in which AB is divided in the ratio $k:l$.

(b) Note that in Case (i) AB , the given line, represents $k + l$ units, but in Case (ii) only $k - l$ units; $\therefore$ the distance along AO from A to the point Y or Z , which is joined to B , is $k + l$ in internal division, and $k - l$ in external division.

(c) What modification of the second construction must be made if $k < l$? Where will O then be? Outside B or outside A ?



EXAMPLES 80.

PRACTICAL.

(The line or segments obtained should be measured, and some of the results checked by arithmetic.)

1. Construct a fourth proportional to

(i) 2.1, 1.5, 1.4; (ii) 4.1, 1.7, 3.

2. Construct a third proportional to (i) 1.7, 2.2; (ii) 4.6, 3.1.

3. Work out, by drawing, the following results

(i) $\frac{1.2 \times 1.7}{1.4}$; (ii) $\frac{2.2^2}{1.9}$; (iii) $\frac{1.9}{2.4}$ (i.e. $\frac{1.9 \times 1}{2.4}$); (iv) $\frac{1}{2.8}$.

4. Divide a line 4'' long internally in the ratio 3 : 2.

5. Divide a line 5.8 cm. long internally in the ratio 4 : 7.

6. Divide a line 2'' long externally in the ratio 5 : 2.

7. Divide a line 4 cm. long externally in the ratio 9 : 4.

8. Divide a line 2.7'' long externally in the ratio 41 : 8.

9. Draw a line 5 cm. long and divide it internally and externally in the ratio (i) 10 : 3; (ii) 5 : 11.

10. Divide a line 4.5'' long into three segments which are as 3 : 2 : 4.

11. Divide a line 5'' long into three parts which are proportional to 5, 3, 1.

12. If one rectangle measures 2.5 cm. by 4.3 cm. and an equal rectangle stands on a base of 3.8 cm., find the height of the latter by drawing.

*If truthe maie trie it selfe
By Reasons prudent skyl,
If reason maie prevayle by right
And rule the rage of will,
I dare the triall byde
For truthe that I pretende,
And though some lyst at me repine,
Juste truthe shall me defende.*

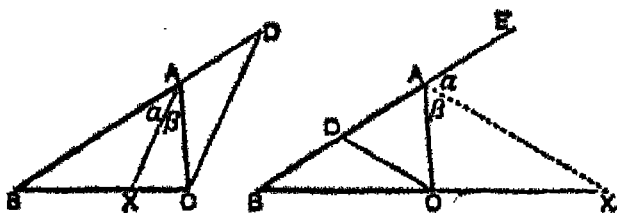
(From Robert Recorde's *Geometry*—"The Pathway to Knowledge," 1551.)

THEOREM 57.

The bisector (internal or external) of an angle of a triangle divides the opposite side (internally or externally) in the ratio of the sides containing the angle bisected; and conversely.

Given a ΔABC in which AX bisects $\angle BAC$ (*internally* in Fig. i and *externally* in Fig. ii), cutting BC or BC produced in X .

To prove that $\frac{BX}{XC} = \frac{BA}{AC}$.



Constr. Label the two equal angles at A α and β .

Draw $CD \parallel AX$ to meet BA or BA produced in D .

Proof. Then $\angle \alpha = \angle ADC$, (corresp.)
 and $\angle \beta = \angle ACD$. (alt.)
 But $\angle \alpha = \angle \beta$ (given)
 $\therefore \angle ADC = \angle ACD$;
 hence in ΔACD , $AD = AC$. (opp. $\angle$ s)

Now $AX \parallel DC$, $\therefore \frac{BX}{XC} = \frac{BA}{AD}$;

but $AD = AC$, (proved)
 $\therefore \frac{BX}{XC} = \frac{BA}{AC}$.

Q.E.D.

Conversely. If the base of a Δ (BC) is divided (internally or externally) (at X) in the ratio of the other sides (BA:AC), then the line from the vertex to the point of section (AX) bisects the vertical $\angle$.

For with the same constr. as before we have

$$\frac{BX}{XC} = \frac{BA}{AC} \text{ (given), and } \frac{BX}{XC} = \frac{BA}{AD} \text{ (by } \parallel \text{)};$$

$$\therefore AC = AD, \therefore \angle ADC = \angle ACD.$$

$$\text{But } \angle ADC = \angle \alpha, \quad (\text{corresp.})$$

$$\text{and } \angle ACD = \angle \beta, \quad (\text{alt.})$$

$$\therefore \angle \alpha = \angle \beta.$$

Q.E.D.

EXAMPLES 81.

NUMERICAL.

1. The sides of a triangle are 4'', 5'', 6''; find the lengths of the parts into which the longest side is divided by the bisector of the opposite angle.

2. ABC is a Δ in which AB = 3'', BC = 4''; if the external bisector of $\angle A$ meets BC produced at a point $3\frac{1}{2}$ '' from C, find AC.

3. In ΔABC , AB = 2'', BC = 3'', CA = 4''; if the bisectors of $\angle A$, internal and external, meet BC, or BC produced, in X, Y, calculate XY and show that $\frac{1}{CX} + \frac{1}{CY} = \frac{2}{CB}$.

4. In ΔABC , BC = 4'', CA = 6'', AB = 9''; if the bisectors of $\angle A$ meet BC, or BC produced, in X, Y, and O is the mid-point of BC, show that $OX \cdot OY = OB^2$.

THEORETICAL.

5. AD is a median of ΔABC ; the bisectors of the $\angle$ s ADB, ADC meet AB, AC in X, Y; prove XY to be parallel to BC.

6. ABED is a parallelogram whose diagonals meet in O, and the bisector of $\angle AOB$ meets AB in X and CD in Y; also the bisector of $\angle AOD$ meets AD in P and BC in Q. Prove that PX and QY are parallel.

7. O is a point inside ΔABC ; the bisectors of angles AOB, AOC meet AB, AC in P, Q; if PQ and BC are parallel, show that O lies on the perpendicular bisector of BC.

8. The four points A, B, C, D on a straight line are such that BC and CD subtend equal angles at P, a point on the circle described on AC as diameter; prove that $AB \cdot CD = BC \cdot AD$.

9. PQR is a Δ right-angled at P, and S a point in QR; find a point T in QR or RQ produced such that $RS:RT = PS:PT$, and prove that each of these ratios = $QS:QT$.

10. In the quadrilateral ABCD, the bisectors of angles A and C meet on BD; prove that $AB \cdot CD = BC \cdot AD$.

11. State and prove the converse of Ex. 10.

12. If I is the incentre of ΔABC and AI meets BC in D, prove that $AI:ID = AB + AC:BC$.

13. O is a point inside ΔABC and the bisectors of angles BOC, COA, AOB meet BC, CA, AB in X, Y, Z. Prove that

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$

14. ABCD is a square and AC, BD meet in O; if the bisector of angle BAO meets BO in F and BC in G, prove that $BF = BG$ and $FO = \frac{1}{2}GO$.

15. AB is a given line and P moves so that $\frac{PA}{PB} = \frac{2}{1}$; mark points X, in AB, and Y, in AB produced, where AB is divided internally and externally in the ratio 2:1. Prove—

- (i) that PX and PY are the bisectors of $\angle APB$;
- (ii) that XPY is a right $\angle$; and state—
- (iii) the locus of point P.

AN IMPORTANT LOCUS—

The Circle of Apollonius.

The locus of P, which moves so that $PA:PB$ is a given ratio, A and B being two fixed points, is a circle.

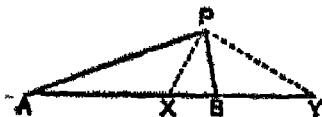


FIG. 306.

Let the given ratio be $k:1$.

Let X and Y be the points where AB is divided internally and externally in the ratio $k:1$;

then X and Y are fixed positions of P.

Also $\frac{AX}{XB} = \frac{k}{1} = \frac{PA}{PB} \therefore PX$ bisects $\angle APB$;

and $\frac{AY}{YB} = \frac{k}{1} = \frac{PA}{PB} \therefore PY$ bisects $\angle APB$ externally.

$\therefore \angle XPY$ is a rt. $\angle$, (half a str. $\angle$)
and, as X and Y are fixed points,
the locus of P is the circle on XY as diameter.

Q.E.D.

Note that the diameter XY , which determines the circle, joins the two points where AB is divided internally and externally in the given ratio.

EXAMPLES 82.

NUMERICAL.

1. AB is a line 3" long; prove that the locus of a point P which moves so that $PA = 2 PB$ is a circle of radius 2" whose centre is 1" from B .

2. AB is a line 20" long and $PA = 1\frac{1}{2} PB$. Show that the locus of P is a circle of radius 24".

PRACTICAL.

3. Draw ΔABC on a base BC of 2" such that $AB = 2AC$ and such that the median bisecting BC is 1.8". Measure AB and AC .

4. Draw a triangle on a base of 3", with an altitude of .8", and having the sides about the vertex in the ratio 3:1. Measure these sides.

5. In a circle of radius 1.4" inscribe a Δ having its base 2.5" long and the ratio of the other sides 2:3. Measure these sides.

6. Draw a triangle with a base 4 cm., a vertical angle of 45° , the ratio of the sides about the angle being 5:2. Measure these sides.

7. Construct a right-angled triangle, the ratio of the sides about the right angle being 3:1 and the hypotenuse being 2.5".

Note. Apollonius of Perga, after whom the circle mentioned above and Theorem 82 are named, lived in the third century B.C. He lectured at Alexandria in the schools of the successors of Enclid, and was known in his own day as the Great Geometer.

SIMILAR FIGURES.

We have already seen in the early part of this book that if one figure is an exact copy of another on a different scale (such as two drawings of a school playground on different scales), two things are necessary:

- (i) *the angles of one figure must be equal to the angles of the other, taken in order;*
- (ii) *the corresponding sides of the two figures are proportional.*

Such figures are said to be similar. They have the *same shape* but not the *same size*. One of the above conditions without the other is not sufficient.

Show that this is so by considering—

- (i) a rectangle and a square;
- (ii) a rectangle cut in two by a line parallel to one side;
- (iii) a large square and a small rhombus;
- (iv) any polygon with a piece cut off by a line parallel to one side. (See Fig. 267.)



FIG. 267.

What happens in the case of two triangles?

- (i) If two triangles have angles of 50° , 60° , 70° , and the base of one is twice the base of the other, what about the ratios of the other pair of sides?
- (ii) If triangles have sides (a) 4 cm., 5 cm., 6 cm., and (b) 4", 5", 6", are their corresponding angles equal?

Take the triangle shown in figure.

How can it be copied on a different scale—

- (i) using sides as much as possible?
- (ii) using angles as much as possible?
- (iii) using sides and angles?

In each case the triangle and its copy are similar.

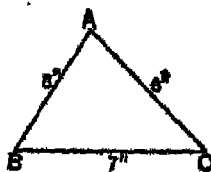


FIG. 268.

This suggests that two triangles have the same shape—

- (i) *if the angles of one equal the angles of the other;*
- or (ii) *if the sides of one are proportional to the sides of the other;*
- or (iii) *if one angle of one triangle = one angle of another triangle, and the sides about these angles are proportional.*

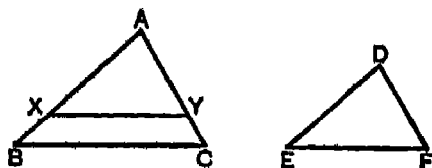
THEOREM 58.

If two triangles are equiangular, their corresponding sides are proportional.

Given two Δ s ABC, DEF, in which

$$\angle A = \angle D, \angle B = \angle E, \angle C = \angle F.$$

To prove that $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$.



Constr. Apply Δ DEF to Δ ABC so that $\angle D$ is placed on the equal $\angle A$, with DE lying along AB and DF lying along AC; let the new position of Δ DEF be AXY.

Proof. Because $\angle AXY = \angle E$ (*constr.*) = $\angle B$ (*given*)

$$\therefore XY \parallel BC,$$

$$\therefore \frac{AB}{AX} = \frac{AC}{AY}.$$

But $AX = DE$ and $AY = DF$,

$$\therefore \frac{AB}{DE} = \frac{AC}{DF}.$$

Similarly it can be proved that

$$\frac{AC}{DF} = \frac{BC}{EF};$$

$$\therefore \frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}.$$

Q.E.D.

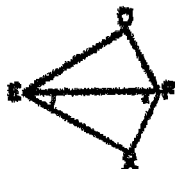
Cor. *Equiangular Triangles are similar.*

THEOREM 59.

If two triangles have their sides proportional, they are equiangular, those angles being equal which are opposite to corresponding sides.

Given two Δ s ABC, DEF, in which $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$

To prove that $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$.



Constr. On the opp. side of EF to D draw a ΔXEF having $\angle XEF = \angle B$ and $\angle XFE = \angle C$.

Proof. Then Δ s ABC, XEF are equiangular;

$$\therefore \frac{AB}{XE} = \frac{BC}{EF}.$$

$$\text{But } \frac{AB}{DE} = \frac{BC}{EF} \quad (\text{given})$$

$$\therefore \frac{AB}{XE} = \frac{AB}{DE},$$

$$\therefore XE = DE.$$

Similarly $XF = DF$.

Then in Δ s DEF, XEF.

$$\begin{cases} DE = XE, & (\text{proved}) \\ DF = XF, & (\text{proved}) \\ EF \text{ is common.} \end{cases}$$

$$\therefore \Delta s \equiv, \quad (3 \text{ sides})$$

so that $\angle DEF = \angle XEF$, which $= \angle B$, (constr.)
and $\angle DFE = \angle XFE$, which $= \angle C$, (constr.)

$$\therefore \angle D \text{ must } = \angle A. \quad (3 \angle s \text{ of a } \Delta)$$

Q.E.D.

Cor. Triangles which have their sides proportional are similar.

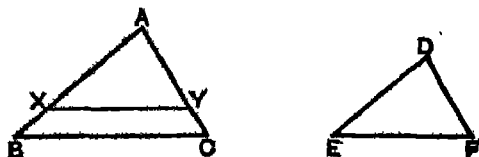
THEOREM 60.

If two triangles have one angle of the one equal to an angle of the other, and also the sides about the equal angles proportional, the triangles are similar.

Given two Δ s ABC, DEF in which $\angle A = \angle D$, and

$$\frac{AB}{DE} = \frac{AC}{DF}.$$

To prove the Δ s ABC, DEF similar.



Constr. Apply Δ DEF to Δ ABC so that $\angle D$ is placed on the equal $\angle A$, with DE lying along AB and DF lying along AC;

let the new position of Δ DEF be AXY.

Proof. $\frac{AB}{DE} = \frac{AC}{DF}$, (given)

but DE = AX and DF = AY,

$$\therefore \frac{AB}{AX} = \frac{AC}{AY};$$

$\therefore XY \parallel BC.$ (sides div^d prop^{ly})

$\therefore \angle B = \angle AX Y,$ (corresp.)
which = $\angle E.$ (constr.)

Similarly $\angle C = \angle F.$

$\therefore \Delta$ s ABC, DEF are equiangular,
the Δ s are similar.

(REMARKS.)

Q.E.D.
R

EXAMPLES 82.

NUMERICAL.

1. A man 6' high casts a shadow 10' 5" long: how high is a house which casts a shadow 35' long?

2. A $\triangle$ has its sides 21 yds., 22 yds., 23 yds. long: it is drawn to scale so that the longest side is 7.5 cm. Find the lengths of the other sides.

3. In a $\triangle ABC$, XY is drawn parallel to AB and meets AC in X and BC in Y .

(i) If $AC = 3''$, $AB = 3.5''$, $BC = 4.5''$ and $XY = 1.4''$, find CX and YS .

(ii) If $CX = 1.3''$, $XA = 1.6''$, $CB = 12''$, $AB = 6.5''$ find XY and YS .

4. A walking-stick 3' 8" long when held upright exactly covers up the height of a lamp-post which is 50' feet away and is 11' high. How far away from me must the walking-stick be held?

5. My eye is 5' 6" above the ground and as I look out of my window I find that the bottom of the window frame, which is 27 $\frac{1}{2}$ " above the floor, just covers a small stone on the ground outside which is 2' 6" from the window. How far am I standing from the window?

6. The diameter of the base of a cone is 10 cm., and a section of it is cut off by a plane which is 3 cm. from the base. If the diameter of this section is 7.5 cm., find the height of the cone.

7. A window measures 8' 6" high and 3' 9 $\frac{1}{2}$ " wide; and is divided up into equal panes which are similar in shape to the window. If the height of each pane is 15.6", how many panes will there be? (Neglect the wooden divisions between the panes.)

8. Two $\triangle$ s ABC , XOY are placed so that their bases BC , OY form a straight line BOY , also $AB = 8''$, $AC = 4''$, $BC = 6''$, $XO = 4''$, $OY = 3''$, $XY = 2''$. Prove that $AB \parallel XO$ and calculate where AX and BY meet when produced.

9. Two $\triangle$ s have their sides 8'', 7'', 9" and 18'', 14'', 21" long. Prove that whichever pair of sides are taken in the first triangle, a pair can be taken in the other having the same ratio. Are the triangles similar? If not, why not?

10. ABC is an angle whose arms BA , BC are produced to X and Y ; if $BA = AX = 3''$, $BC = 2''$, $CY = 7''$, prove that $\triangle$ s ABC , XBY are similar. Which angle of $\triangle XBY$ equals $\angle BAC$?

11. A man 5' 9" high stands 10' 6" away from a lamp-post 13' 9" high; how long is the shadow of the man which is thrown on the ground by the lamp at the top of the post?

12. I see that the top of a post 10' 6" high is directly in a line with the top of a distant building 15' 6" high which stands 13' away from the post. If my eye is 5' 6" above the ground, how far am I from the building?

THEORETICAL.

13. Two isosceles triangles are similar if their vertical angles are equal.

14. The line joining the mid-points of two sides of a triangle is half the third side.

15. A trapezium has one of its parallel sides twice the other; prove that the point of intersection of its diagonals is a point of trisection of each diagonal.

16. $\angle AOB$ is an angle, AA' is drawn $\perp OB$ and $BB' \perp OA$. Fill in the blanks in the statements $\frac{AA'}{BB'} = \frac{OA}{OB} = \frac{OA'}{OB'}$.

17. AB and CD are chords of a circle intersecting at O ; prove that $\triangle AOC$, $\triangle BOD$ are similar, and fill in the blanks in the statements $\frac{AO}{BO} = \frac{OC}{OD} = \frac{AC}{BD}$.

18. OAB and OCD are two secants of a circle; prove that $\triangle OAD$, $\triangle OBC$ are similar and complete the statements $\frac{OA}{OB} = \frac{OD}{OC}$. Deduce that $OA \cdot OB = OC \cdot OD$.

19. From a point O a tangent OT and a secant OAB are drawn to a circle. Express the statement $OT^2 = OA \cdot OB$ in the form of an equality of ratios, and prove this equality by similar triangles.

20. If AD and BE are altitudes of $\triangle ABC$, prove $\frac{AD}{BE} = \frac{AC}{BC}$.

21. Prove that the external common tangents of two circles divide the line of centres externally in the ratio of the radii. Consider also the case of the internal tangents, when such exist.

22. If three concurrent straight lines AX , BY , CZ are cut by two parallel transversals in A , B , C and X , Y , Z show that $\frac{AB}{XY} = \frac{BC}{YZ}$

(Is this statement equivalent to $\frac{AB}{BC} = \frac{XY}{YZ}$? Which is the easier to prove?)

23. Prove that a line drawn parallel to the base of a triangle and terminated by the other two sides is bisected by the median drawn to the base.

24. Two triangles ABQ , DBC equal in area are drawn on the same base BQ and on the same side of it. If a line parallel to BQ cuts AB , AC , DB , DC in P , Q , R , S , prove that $PQ = RS$.

25. In $\triangle ABC$ the side $AB = 3''$, $BC = 4''$, $CA = 2''$; AD is the tangent at A to the circumference of the triangle, and D is taken on it on the opposite side of AB to C so that $AD = 6''$. Prove that $\triangle ABD$, ABC are similar and that BC touches the circle through A , B , D . Calculate BD .

26. ABC and DEF are two similar triangles; if corresponding medians AX , DY are drawn, prove that $AB:DE = AX:DY$.

27. If AD , BE are medians of $\triangle ABC$ meeting in G , prove that $AG = \frac{2}{3}GD$ and $BG = \frac{2}{3}GE$. Hence show that the three medians of a triangle meet in a point.

28. AOB is a diameter of a circle, and the tangents AD , BO at A and B are met by any third tangent at D , C . Prove that $\triangle AOD$, BOC are similar, and deduce that $AD \cdot BC = (\text{radius})^2$.

29. If PN is the perpendicular from P , a point on a circle, to the tangent at another point A , and AB be the diameter through A , prove that $AP^2 = AB \cdot PN$. [Arrange this equality in the form of two equal ratios, and so find out which triangles need to be proved to be similar.]

30. ABO is an equilateral triangle inscribed in a circle; P is any point on the minor arc AC ; AP , BO are produced to meet in Q , and AO , BP meet in O . Prove that $\triangle AOB$, ABQ are similar.

31. The base BC of an isosceles triangle ABO is produced both ways to D and E , D being beyond B . If $AB^2 = DE \cdot OE$, prove that $\triangle EAO$, ADB are similar. Which are the pairs of equal angles?

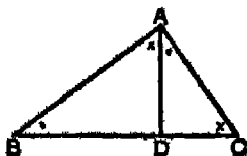
32. $ABCD$ is a para^m and BC is bisected at P and CD at Q . If AP , AQ meet BD in X , Y , prove that X and Y trisect BD . [Either (i) prove $BX = \frac{1}{3}XD$ by similar triangles, or (ii) join AC , PQ .]

THEOREM 61.

If a perpendicular is drawn from the right angle of a right-angled triangle to the hypotenuse, the triangles on each side of the perpendicular are similar to the whole triangle and to one another.

Given a $\triangle ABC$ having $\angle A$ a right angle, and the perp. AD drawn from A to the hypotenuse BC .

To prove that $\triangle s ABD, ADC, ABC$ are all similar.



Proof.

In $\triangle s ABD$ and ABC ,
 $\angle B$ is common,
 and $\text{rt. } \angle ADB = \text{rt. } \angle BAC$,
 $\therefore \triangle s$ are equiangular.

Sim^{ly} it may be proved that $\triangle s ADC, ABC$ are equi $\angle$;
 $\therefore$ the three $\triangle s$ are equiangular, and $\therefore$ similar.

Q.E.D.

Cor. 1. *The square on the perpendicular is equal to the rectangle contained by the segments of the hypotenuse.*

For $\triangle s ABD, CAD$ are similar,
 $\therefore \frac{BD}{DA} = \frac{DA}{DC} \therefore AD^2 = BD \cdot DC$.

Cor. 2. *The square on either of the sides containing the right angle is equal to the rectangle contained by the hypotenuse and the segment of the hypotenuse adjacent to that side.*

For $\triangle s ABD, CBA$ are similar,
 $\therefore \frac{BD}{BA} = \frac{BA}{BC} \therefore BA^2 = BD \cdot BC$.

So also $CA^2 = CD \cdot CB$.

Cor. 3. By adding, we get

$$BA^2 + CA^2 = BC(BD + CD) = BC^2.$$

This gives us a very brief proof of Pythagoras.

Note (a).—Cor. 1 can also be proved by two intersecting chords, and Cor. 2 by tangent and secant.

Note (b).—Corollaries 1 and 2 show us three Mean Proportionals in the figure of Theorem 51:

thus DA is the mean prop. to BD and DC,
 BA " " " BC and BC,
 CA " " " CD and CB.



FIG. 269.

In the first of these the lines (*a*, *b*) to which a mean prop. (*m*) is found do not overlap; in the second and third the lines (*a*, *d*) or (*e*, *f*) do overlap, and their mean proportionals are *n* and *p*.

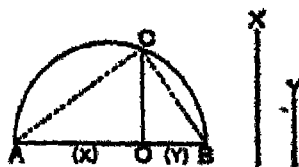
PROBLEM 17.

To find a mean proportional to two given lines.

Let *X* and *Y* be the two given lines.

To find a mean prop^l to *X* and *Y*.

METHOD I. (*Non-overlapping method*).



Place *X* and *Y* in a straight line (*AO*, *OB*) so as not to overlap;

on *AB* as diam. describe a semicircle;

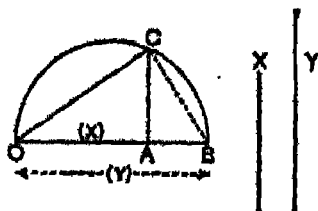
at *O* erect *OC* $\perp$ *AB* to meet the circle in *C*;

join *CA*, *CB*.

Then since $\angle ACB$ is a rt. $\angle$ and $OC \perp$ hypot. *AB*,
 $OC^2 = AO \cdot OB$ and *OC* is a mean prop^l. to *X* and *Y*.

Q.E.F.

METHOD II. (*Overlapping method*). Suppose $y > x$.



Place X and Y in a straight line OA , OB so as to overlap;

on the line OB as diam. describe a semicircle,
and at A erect $AO \perp OB$ to meet the circle in C ;

join CO , CB .

Then since OCB is a rt. $\angle$ and $OA \perp$ hypot. OB ,

$CO^2 = OA \cdot OB$ and CO is a mean prop^l. to X and Y .

Q.E.F.

EXAMPLES 84.

PRACTICAL.

1. Find a mean proportional to lines of (i) 3", 1.6"; (ii) 5", 3"; (iii) 7 cm., 4 cm.
2. Draw lines $\sqrt{2} \times 1.3$ inches and $\sqrt{4} \times 2.5$ inches long.
3. Draw a line whose length is $\sqrt{7}$ ". (Take $7 = 2 \times 3.5$ rather than 7×1 . Why?)
4. Draw lines of length $\sqrt{11}$ ", $\sqrt{5}$ ", $\sqrt{32}$ cm., $\sqrt{10}$ cm., $\sqrt{21}$ ".
5. Draw a line 3" long, and divide it into segments whose mean proportional is 1.2".
6. Draw a line AB ; find a point C in it such that $AC^2 = \frac{1}{4}AB^2$.

THEORETICAL AND NUMERICAL.

7. $\angle A$ in $\triangle ABC$ is 90° and $AD \perp BC$. If $AB = 3$ ", $BC = 4$ ", find BD , DC .

8. In $\triangle ABC$ $\angle C$ is a right $\angle$; OD is the perpendicular to AB , and CE , the bisector of $\angle C$, meets AB in E . If $AC = 9$ ", $BC = 12$ ", calculate the length of DE .

9. The rectangle contained by the hypotenuse of a right-angled triangle and the corresponding altitude is equal to the rectangle contained by the other two sides of the triangle.

10. If AD is an altitude of $\triangle ABC$ and $AD^2 = BD \cdot DC$, prove that the angle BAC is a right angle.

11. If AD $\perp$ base BC of $\triangle ABC$ and BA is a mean proportional to BD and BC, show that $\angle BAC$ is a right angle.

ALTERNATIVE PROOFS OF THEOREMS 53 AND 54 (PAGES 215 AND 216) BY MEANS OF SIMILAR TRIANGLES.

THEOREM 53.

If two chords of a circle intersect at a point inside the circle, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

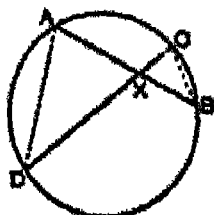


FIG. 270.

Given two chords of a circle, AB and CD, intersecting at X.

To prove that $AX \cdot XB = CX \cdot XD$.

Constr.

Join AD, BC.

Proof.

Then in $\triangle AXD$, BXC ,

$\angle A = \angle C$, (*subtended by same arc*)

and $\angle AXD = \angle BXC$, (*vert. opp.*)

$\therefore \triangle$ s are equiangular, and $\therefore$ similar;

$$\therefore \frac{AX}{CX} = \frac{XD}{XB}$$

$$\therefore AX \cdot XB = CX \cdot XD.$$

THEOREM 54.

If from a point outside a circle a secant and a tangent to the circle are drawn, the rectangle contained by the whole secant and the segment of it outside the circle is equal to the square on the tangent.

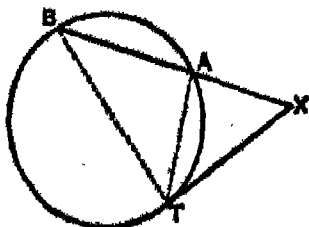


FIG. 271.

Given a circle ABT with a secant XAB and a tangent XT drawn to it from the external point X .

To prove that $XB \cdot XA = XT^2$.

Constr. Join AT , BT .

Proof. In Δs XAT , XTB ,

$\angle X$ is common, and $\angle ATX = \angle B$, (in alt. segment)

$\therefore \Delta s$ are equiangular, and $\therefore$ similar,

$$\therefore \frac{XA}{XT} = \frac{XT}{XB},$$

$$\therefore XB \cdot XA = XT^2.$$

Cor. If two chords of a circle intersect at a point outside the circle, the rectangle contained by the segments of one is equal to the rectangle contained by the segments of the other.

Thus in the figure $XB \cdot XA = XD \cdot XC$, for both $= XT^2$.

Or this can be proved directly by the method of Theorem 53, the proof being as given on p. 248, except that $\angle s$ AXD , BXC are not vertically opposite, but the same angle.

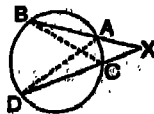
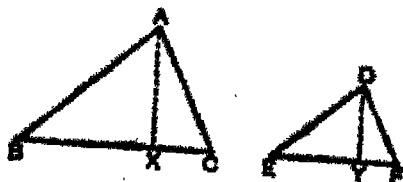


FIG. 272.

Note. Examples on Theorems 53 and 54 will be found on pages 218-220.

THEOREM 62.

The areas of similar triangles are proportional to the squares on corresponding sides.



Given two similar Δ s ABC, DEF, BC and EF being corresponding sides.

To prove that $\frac{\Delta ABC}{\Delta DEF} = \frac{BC^2}{EF^2}$

Constr.

Draw AX, DY $\perp$ BC, EF.

Proof.

$$\begin{aligned} \frac{\Delta ABC}{\Delta DEF} &= \frac{\frac{1}{2} BC \cdot AX}{\frac{1}{2} EF \cdot DY} \quad (\Delta = \frac{1}{2} \text{ base} \times \text{alt.}) \\ &= \frac{BC \cdot AX}{EF \cdot DY} \end{aligned}$$

Now in Δ s ABX, DEY,
 $\angle B = \angle E$ (given), and $\angle X = \angle Y$ (rt. $\angle$ s),
 $\therefore \Delta$ s are equiangular and $\therefore$ similar,

$$\therefore \frac{AX}{DY} = \frac{AB}{DE},$$

$$\text{which} = \frac{BC}{EF};$$

(given)

$$\therefore \frac{\Delta ABC}{\Delta DEF} = \frac{BC \cdot BC}{EF \cdot EF} = \frac{BC^2}{EF^2}.$$

Q.E.D.

Cor. If Δ s ABC, DEF have $\angle B = \angle E$,

$$\text{then } \frac{\Delta ABC}{\Delta DEF} = \frac{BC \cdot AB}{EF \cdot DE}.$$

This follows from the first six lines of the above proof.

EXAMPLES 85.

NUMERICAL.

1. Two similar triangles have bases of 1.4" and 2.1". The area of the first is 2.4 sq. ins. Find the area of the other.
2. The areas of two similar triangles are 48 and 75 sq. ins. If the base of the first is 2.4", find the base of the other.
3. Two similar triangles have areas of 3.25 and 4.41 sq. cm. Find the ratio of (i) their bases; (ii) their altitudes.
4. The sides of a triangle are 3", 3.5", 4". What are the sides of a similar triangle whose area is nine times as great?
5. Two triangles are drawn with their sides (i) 2", 3", 4"; (ii) 3', 3', 4'; find the ratio of their areas.
6. $\triangle ABC$ is bisected by a line XY which is parallel to BC . Find the ratio $AX : XB$ correct to two decimal places.

THEORETICAL.

7. Prove that the areas of similar triangles are proportional to the squares of their altitudes.
8. Show that the areas of similar triangles are proportional to the squares of corresponding medians.
9. From a point O outside a circle two secants OAB , OCD are drawn. If chord $BC = 2 \times$ chord AD , prove $\triangle OBC = 4 \triangle OAD$.
10. ABC is a triangle; AB is trisected at P and Q , and AC is trisected at R and S , P and R being the points of trisection nearer to A . Prove that area of trapezium $PRSQ$ is $\frac{1}{3}$ of that of $\triangle ABC$.
11. The sides BC , CA , AB of a $\triangle ABC$ are divided into five equal parts; P and Q are the first and fourth points of division on BC , R and S on CA , T and V on AB . Prove that area of $PQRSTV = \frac{1}{25}$ of $\triangle ABC$.
12. ABC , DEF are similar triangles, and on their bases BC , EF are described equilateral triangles PBC , QEF . Prove that $\triangle ABC : \triangle DEF = \triangle PBC : \triangle QEF$.
13. Two chords AB , CD of a circle meet in O . Prove that $\triangle AOC : \triangle BOD = AO^2 : OD^2$.
14. In the figure to Theorem 61, prove that $EA^2 : AC^2 = BD : DC$.
15. $ABOD$ is a rectangle and AX , OY are drawn at right angles to BD ; if BD is thus trisected, prove that the length of the rectangle is $\sqrt{2}$ times its breadth.

16. $\triangle ABC$ is a triangle; X bisects AB and Y is a point on AC such that $AY:YC = 2:1$. Find the value of the ratio $\triangle AXY : \triangle ABC$.

17. $\triangle ABC$ is a triangle; P, Q, R are points on BC, CA, AB such that $BP = \frac{1}{3}BC, CQ = \frac{1}{3}CA, AR = \frac{1}{3}AB$. Prove that

(i) $\triangle BPR = \frac{1}{9}\triangle ABC$; (ii) $\triangle PQR = \frac{1}{9}\triangle ABC$.

18. In $\triangle ABC$ points P, Q, R are taken on BC, CA, AB so that $BP:PC = CQ:QA = AR:RB = 2:1$. Prove that

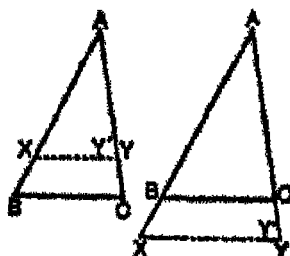
$$\triangle PQR = \frac{1}{9}\triangle ABC.$$

19. In $\triangle ABC$ the sides AB, AC are produced to D, E , and DF is drawn $\parallel BE$ to meet AC produced in F . Prove that

$$\triangle ABC : \triangle ADE = AC : AF.$$

ALTERNATIVE PROOF OF THEOREM 56 (PAGE 239).

Constr. Draw $XY' \parallel BO$ to meet AC in Y' .



$$\text{Then } \frac{AX}{XB} = \frac{AY'}{Y'O}; \quad (XY' \parallel BO)$$

$$\text{but } \frac{AX}{XB} = \frac{AY}{YO}; \quad (\text{given})$$

$$\therefore \frac{AY}{YO} = \frac{AY'}{Y'O};$$

i.e. AC is divided in the same ratio at Y and Y' both internally or both externally,

$\therefore Y$ and Y' coincide.

But $XY' \parallel BO$,

$\therefore XY \parallel BO$.

Q.E.D.

A MENTAL (OR ORAL) TEST IN GEOMETRY.

Examples 1-17 can be taken at the end of Section I, 18-25 at the end of Section II, 26-31 after Section IV, 32-40 after Section V, and the rest at the end of the section on Similar Triangles.

1. How many sides has (i) a pentagon, (ii) a hexagon, (iii) an octagon?

2. How many edges has (i) a cube, (ii) a triangular pyramid, (iii) a square prism? Make a sketch of each.

3. How many faces has (i) a cube, (ii) a rectangular pyramid, (iii) a hexagonal prism? Sketch each.

4. If AB crosses CD at O and $\angle AOB = 85^\circ$, find the other angles.

5. In Fig. 294 what is the name given to angles (i) α and β , (ii) α and γ , (iii) β and γ ?

6. If in Fig. 294 AB and CD are $\parallel$, and α is 42° , find β , γ , δ , θ .

7. If three lines meet at O , what is the sum of the three angles at O ?

8. In $\triangle ABC$, $\angle A = 39^\circ$, $\angle B = 73^\circ$, find $\angle C$.

9. In $\triangle ABC$, BC is produced to D ; $\angle ACD = 119^\circ$ and $\angle A = 53^\circ$; find $\angle B$.

10. In a quad^l $ABCD$, $\angle A = 75^\circ$, $\angle B = 88^\circ$, $\angle C = 142^\circ$ find $\angle D$.

11. What sort of quad^l is the figure in Ex. 10?

12. In the case of a figure of eight sides, how many degrees are there in the sum of (i) all the exterior angles, (ii) all the interior angles?

13. If each angle of a polygon is 170° , how many sides has it?

14. What is the size in degrees of each (i) exterior angle, (ii) interior angle of a regular figure of (i) 10 sides; (ii) n sides?

15. In a parm one side is $5''$ and one angle is 72° ; what do you know about the other sides and angles?

16. What name can be given to a quad^l if its diagonals (i) bisect each other; (ii) bisect each other at right angles; (iii) bisect each other and are equal; (iv) cut at right angles and one only is bisected by the other?

17. The sides of a triangle are $6''$, $7''$, $8''$. State (i) the direction, (ii) the length of the line joining the mid points of the first two sides.

18. Find the area of a parm on $4''$ base and with height $2\frac{1}{2}''$.

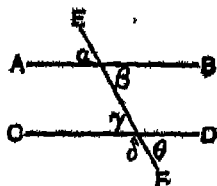


FIG. 294.

19. The areas of two triangles are equal; one has a base of $2''$ and a height of $4\frac{1}{2}''$; the other has a base of $3''$; find its altitude.

20. A para^m and a triangle are equal in area; their bases are equal. What can you say about their heights?

21. Find the area of $\triangle ABC$ if $\angle B = \angle C = 45^\circ$ and $AB = 5'$.

22. Find the side of a square if its area = that of a rectangle with adjacent sides $4\frac{1}{2}''$ and $8''$.

23. Find the third side of a right-angled triangle if the longest side is 17 cm, and the shortest is 8 cm.

24. The sides of a triangle are $21''$, $28''$, $35''$. Is it right-angled?

25. The sides of a trapezium are $5''$ and $13''$, the other sides are $5''$ and $13''$, and the angles at each end of the $5''$ line are right angles. Find the area of figure.

26. A chord of a circle is $24''$ long and the radius is $13''$; find the distance of the chord from the centre of the circle.

27. A $16''$ chord is distant $15''$ from the centre of the circle. Find radius of circle.

28. Find the length of the tangent drawn to a circle of $8''$ diameter from a point $5''$ away from the centre of circle.

29. If two circles have diameters of 7 cm. and 11 cm., how far apart are their centres if they touch (i) externally, (ii) internally?

30. A cyclic quadrilateral has one angle four times the opposite angle, and another angle is 70° . Find all the angles.

31. Find the length of an external common tangent to two circles, whose radii are $2'$ and $7'$, and whose centres are $13'$ apart.

32. Two chords of a circle intersect inside the circle; the segments of one chord are a'' and b'' ; one segment of the other is c'' ; find the remaining segment.

33. Two chords AB, CD of a circle meet outside the circle at O. If AB is $32''$, BO = $4''$, OD = $8''$, find OD; find also the length of the tangent from O to the circle.

34. ACB is a chord of a circle ABD (centre O); if AC = $2''$, CB = $8''$, CO = $3''$ find the radius.

In Fig. 295 complete the statements in Exs. 35-37.

35. $AB^2 = BC^2 + CA^2 + \dots$

36. $AB^2 = AC^2 + 2AC \cdot CE + \dots$

37. $AC^2 = AB^2 + BC^2 - \dots$

38. In Fig. 295 write a statement similar to the above, bringing in the quantity $2AC \cdot AE$.

39. In $\triangle KLM$, $AB \parallel LM$, meeting KL in A and KM in B . If $KA = 3''$, $AL = 4''$, $KB = 5''$, find BM .

40. In the figure to Ex. 39, if $KA = 5''$, $AL = 10''$, $AB = 4''$, find LM .

41. Equiangular figures are not necessarily similar. Give examples. What about the case of triangles?

42. In the figure to Ex. 39, if $KA = 3$ cm., $AL = 7$ cm., what is the ratio of area of $\triangle KAB$: area of $\triangle KLM$?

43. In $\triangle PQR$, $\angle P = 90^\circ$ and $PX \perp QR$. What is the mean proportional between (i) QX and XR , (ii) QX and QR ? What lines is PR the mean proportional between?

44. α and β are similar $\triangle$ s. The sides of α are $5''$, $6''$, $7''$; the smallest side of β is $2''$; find the other sides.

45. α and β are similar triangles having areas of 16 and 49 sq. cm. If the base of α is 12 cm., find the base of β .

46. Calculate x in Fig. 296.



Fig. 296.

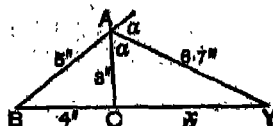


Fig. 297.

47. Calculate x in Fig. 297.

Complete the statements in Nos. 48-53 referring to the figures mentioned.

48. In Fig. 296 (without the numbers), $\frac{AB}{AC} = \dots$

49. In Fig. 297 (without the numbers), $\frac{AB}{AC} = \dots$

50. In Fig. 298, $\frac{QE}{ER} = \frac{QE}{QR} = \frac{QE}{QR} = \frac{QE}{QR}$, and $\frac{QE}{QR} = \frac{QE}{QR} = \frac{QE}{QR}$.

51. In Fig. 299, $AO \cdot OB = \dots = \dots^2 = \dots^2$.

52. In Fig. 300, $OG \cdot OH = \dots = \dots^2 = \dots^2 = \dots^2$.

53. In Fig. 298, $\frac{\Delta QEF}{\Delta QPR} = \frac{QE^2}{QR^2} = \frac{QE^2}{QR^2} = \frac{QE^2}{QR^2}$.

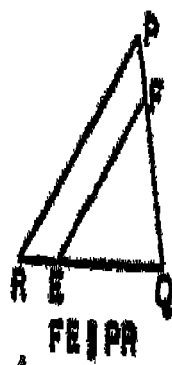


FIG. 298.



FIG. 299.

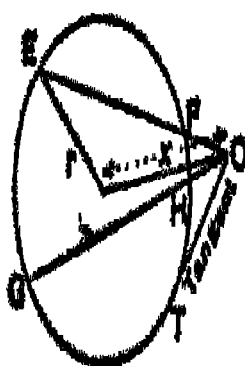


FIG. 300.

REVISION EXAMPLES.

(Angles at a point, parallels, angles of triangle and polygon.)

A. 1. (i) Find the acute angle between the directions N.E. and N. 80° W.

(ii) Name two directions that are perpendicular to S. 25° E.

A. 2. What angle is (i) twice its supplement?

(ii) four times its complement?

A. 3. A line AB is 3 in. long. O is a point in AB such that $AC = 4$ CB, and AB is produced to D so that $AD = 3$ BD. How long are AC, AD, CD?

B. 1. In Fig. B 1 arrows indicate parallel lines. Find x and y in degrees.

B. 2. The sides of a quadrilateral are produced in order, and three of the exterior angles are 70° , 85° , 92° . Find the fourth exterior angle and all the interior angles.

B. 3. In Fig. B 3, the angles x are both 72° ; find $\angle y$ and name two lines through O that are in the same straight line.

C. 1. In $\triangle ABC$, $\angle B$ is twice $\angle A$, and $\angle C$ is three times $\angle B$. Find the angles in degrees.

C. 2. Find the number of sides a polygon has :—
(i) if each exterior angle is 24° ;
or (ii) if each interior angle is 144° .

C. 3. In $\triangle ABC$, $AB = AC$, and BA is produced to D. Prove that $\angle DAC$ is twice $\angle C$.

D. 1. In $\triangle ABC$, $\angle A = 48^\circ$, $\angle B = 86^\circ$, and AC is produced to D. Find $\angle AXD$ if AX bisects $\angle A$ and CX bisects $\angle BCD$.

D. 2. In Fig. D 2 AB and EF are parallel; find $\angle x$, giving reasons for your statements.

D. 3. In a polygon the sum of the interior angles equals the sum of the exterior angles; how many sides has the figure?

E. 1. ABC is a straight line and X bisects AB; prove that $AC + BC = 2$ CX.

(SHORTER)

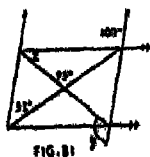


FIG. B.1

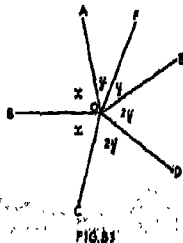


FIG. B.3

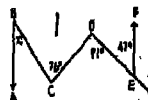


FIG. D.2

E. 2. ABCD is a quadrilateral with $AB \parallel DC$, and $\angle A = \angle B$; prove that $\angle C = \angle D$.

E. 3. ABCDE is a pentagon in which $\angle EAB = 60^\circ$ and $\angle CDE = 130^\circ$, $AB \parallel ED$, and $BC \parallel AE$. Calculate the size of $\angle BCD$.

(Including congruences.)

F. 1. In $\triangle ABC$, $\angle B = 40^\circ$, $\angle A = 72^\circ$; BX , drawn $\perp AC$, and AY , drawn $\perp BC$, meet in O ; find $\angle AOB$.

F. 2. (i) If all the angles of a pentagon are equal, find the size of each.

(ii) Each exterior angle of a polygon equals the interior angle of an equilateral triangle; how many sides has the polygon?

F. 3. In $\triangle ABC$, $AB = AC$, and $\angle BAC$ is obtuse. Equilateral triangles BCD , CAE , ABF are drawn externally on the sides of $\triangle ABC$. Use congruent triangles to prove that $DE = DF$.

G. 1. If the bisectors of angles B and C of $\triangle ABC$ meet in I and $\angle A = 80^\circ$, find $\angle BIC$.

G. 2. In Fig. G 2, prove that $a + b + c + d = 360^\circ$.

G. 3. Two circles have the same centre O , and a straight line $ABCD$ cuts the larger circle in A and D and the smaller one in B and C . Prove by congruent triangles that $AB = CD$.

H. 1. In $\triangle ABC$, $AB = AC$ and XY is drawn $\parallel BC$ to cut AB at X and AC at Y . Prove that $AX = AY$.

H. 2. AB is a straight road, and a man stands off the road so as to be as far from A as he is from B ; if $PN \perp AB$, prove that $AN = NB$.

H. 3. ABCDEF is a hexagon with all its sides equal and all its angles equal; calculate the angle BFD .

I. 1. The angles of a quadrilateral in order are x° , $(x + 20)^\circ$, $(x + 50)^\circ$, $(x + 70)^\circ$. Find x , and show that two sides are $\parallel$.

I. 2. In $\triangle ABC$, $\angle A = 85^\circ$ and $\angle C = 55^\circ$; also $AD \perp BC$. Prove that $AD = BD$.

I. 3. In a rectangle $ABCD$, AB is longer than BA . A point P is taken in DC such that $PC = BC$, and a circle with P as centre and PA as radius cuts CB (or CB produced) in Q . Prove by congruent $\triangle$ s that $QC = DP$.

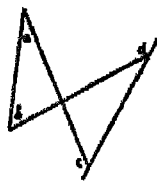


FIG. G 2

J. 1. In the figure to Theorem 3 (p. 59) AOB is straight and $\angle AOD = \angle BOC$; prove that COD is also a straight line.

J. 2. In $\triangle ABC$, O bisects AB and $OA = OC$; prove that $\angle ACB$ is a right angle.

J. 3. In $\triangle s ABC, PQR$, $AB = PQ$ and $BC = QR$; X bisects BC and Y bisects QR and $AX = PY$. Prove that $AC = PR$.

(Including inequalities and parallelograms.)

K. 1. In $\triangle ABC$, O is a point in AC such that $OA = OB$; also $\angle A = 65^\circ$ and $\angle C = 25^\circ$. Prove that $AC = 2 OB$.

K. 2. In $\triangle ABC$, $CA = CB$, $\angle A = 70^\circ$, and BC is produced to D so that $\angle CAD = 30^\circ$. Find the angle ADB and find which of the lengths BC, CD is the greater.

K. 3. ABCD is a quadrilateral with $AB \parallel DC$ and $AD = BC$. (AB is not equal to DC.) Prove, by drawing a parallel to BC through A, that $\angle D = \angle C$.

L. 1. Give in each case the name of a quadrilateral in which—

- (i) two of the sides only are parallel;
- (ii) the diagonals bisect each other;
- (iii) all four sides are equal;
- (iv) the four angles are equal;
- (v) the diagonals are equal and bisect each other;

L. 2. In $\triangle ABC$ the bisectors of the angles A and B meet in I. Show that $\angle AIB = 90^\circ + \frac{1}{2} \angle C$.

L. 3. ABCD is a par^m and X is taken in AB and Y in CD so that $AX = \frac{1}{3} AB$ and $YD = \frac{2}{3} CD$. Show that AXC Y is a par^m.

M. 1. In $\triangle ABC$, $\angle A = 2 \angle B$ and $\angle C = \angle B + 2 \angle A$. Find $\angle B$ in degrees.

M. 2. O is a point inside $\triangle ABC$. The par^m AOB F and AOCE are drawn. Show that $FE = BC$.

M. 3. In $\triangle ABC$, $AB = AC$; BC is produced to D and AD is joined; $\angle B = 65^\circ$, $\angle D = 52^\circ$. Which is the longer: (i) AB or BC; (ii) BD or AD? State the reasons why.

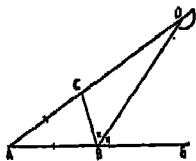


FIG. N1

N. 1. In Fig. N 1 $AB = AC$ and BD bisects $\angle EBC$. Prove that the external angle of $\triangle BCD$ at D $= 3 \angle EBD$.

N. 2. ABCDE is a regular pentagon. EF is drawn perp^r to EA to meet AC produced in F. Calculate $\angle s AEC, AFE, FCD$.

N. 3. ABCD is a par^m; if $\angle DAC > \angle BAC$, prove that $DC > BC$.

O. 1. $ABCD$ is a rectangle and on AB , BC , CD as bases equilateral triangles ABK , BCL , CDM are drawn outside the rectangle. Prove that $KL = LM$.

O. 2. In a pentagon $ABCDE$ of any shape prove that

$$AB + BC + CD + DE > AE.$$

O. 3. Two par^{ms} $PQRA$, $PBBQ$ have one side PB common. Prove that QR bisects AB .

(Including intercepts.)

P. 1. In $\triangle ABC$, X is a point in AB such that $AX = \frac{1}{2} AB$. XY is drawn $\parallel BC$ to meet AC in Y . By drawing another line $\parallel BC$ prove that $AY = \frac{1}{2} AC$.

P. 2. $ABCDE \dots$ is a part of a regular 12-sided polygon. Prove that $AB \perp DE$.

P. 3. In $\triangle ABC$, $AB = AC$ and BC is produced to D ; $DE \perp AB$ and $DF \perp AC$ produced. Prove that BD bisects $\angle EDF$.

Q. 1. In $\triangle ABC$, X bisects AB and Y bisects AC . If Z is any point in BC , prove that XY bisects AZ .

Q. 2. In the quadrilateral $ABCD$, $AB = AD$ and $CB = CD$. If P is any point in AC , prove by two pairs of congruent triangles that $PB = PD$.

Q. 3. Squares $BCYX$ and $BAKH$ are drawn externally on the sides BC , BA of $\triangle ABC$. Prove by congruent triangles that $\angle HCB = \angle AXB$ and deduce that $AX \perp HC$.

R. 1. Prove that, if a straight stick is held at a slant above the ground, the sum of the heights of its ends is twice the height of its mid-point. (Draw a horizontal line through the bottom of the stick or through its mid-point.)

R. 2. In $\triangle ABC$, $AB = AC$; points D and E are taken in BC produced, D beyond B and E beyond C , so that $BD = CE$; on the internal bisector of $\angle A$ a point F is taken. Prove that $DF = EF$. (Produce the bisector of $\angle A$ to meet BC .)

R. 3. In $\triangle ABC$, $AB = AC$, AB is produced to D and AC to E so that $BD = CE$. If BE and CD meet in F , prove that

(i) $\triangle s BCD$, CBE are congruent, and (ii) $\triangle BFC$ is isosceles.

S. 1. $\triangle ABC$ and $\triangle DEC$ are on opposite sides of BC ; $AB = AC$ and $\angle CBD$ is 90° . Prove that a line from A drawn $\perp BC$ will, when produced, bisect CD .

S. 2. $ABOD$ is a par^m and E any point. F bisects AE , and BF is produced to G so that $FG = BF$. Prove that ED and GC bisect each other.

S. 3. In $\triangle ABC$, $AB = AC$; BA is produced to X . If the bisector of $\angle ACB$ meets AB at Y , prove that $\angle CYX = \frac{1}{2} \angle CAX$.

T. 1. In the $\text{par}^m ABCD$, X bisects AB and Y bisects CD . AC meets DX at P and BY at Q . Prove that $AP = PQ = QC$.

T. 2. In the $\text{par}^m ABCD$ the side $AB =$ the diagonal AC and the mid points of BC and DA are E and F . Prove that $AECF$ is a rectangle.

T. 3. In the $\text{par}^m ABCD$, $\angle B = 45^\circ$. Squares $ABEF$, $CDGH$ are drawn outside the par^m and DF , BH are joined. Find the magnitude of $\angle DAF$ and show that $DF = BH = BD$.

(Including areas of parallelograms and triangles.)

U. 1. The diagonals of the quadrilateral $ABCD$ cut at O . If $\triangle AOB$, $\triangle COD$ are equal in area, prove that $AD \parallel BC$.

U. 2. In $\triangle ABC$, $BC = CA$ and $\angle ACB$ is obtuse. BC is produced to D as far as will make $AD = AC$. O is a point in AB produced beyond B such that $AO = AD$. If $\angle ABC = x$, prove that $\angle AOD = 6x - 180^\circ$.

U. 3. In $\triangle ABC$, $AB = 3 BC$; prove that the perpendicular from A to BC is three times the perpendicular from C to AB .

V. 1. The Fig. V 1 shows two sets of three parallel lines. Show that the area of $ABCD$ is half the area of $WXYZ$.

V. 2. $ABCD$ is a par^m . Q is a point in AB , and P is the mid-point of AQ . The parallel through Q to PD meets AD produced at R , and BR cuts CD at X . Use the mid-point theorem to prove that $DX = XC$.

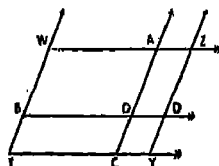


FIG. V.1

V. 3. In Fig. V 3 O bisects AB and XZ ; P bisects AC and XY . What kind of figures are $AZBX$ and $AYCX$? Prove that $BZ = CY$.

W. 1. In Fig. W 1 prove that the sum of the angles marked at A, B, C, D, E is two right angles. (Join CE.)

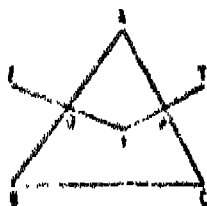


FIG. W 1

W. 2. AB, CD, EF are parallel lines met by two non-parallel lines in A, C, E and B, D, F respectively. Prove, by breaking up each triangle into two, that $\triangle ADE \cong \triangle BCF$ in area.

W. 3. ABCD is a quadrilateral in which $AB = AD$ and $\angle B = \angle D$. Prove that the diagonals AC, BD cut at right angles. Prove also that the area of ABCD is $\frac{1}{2} ab$, where $AC = a$ and $BD = b$.



FIG. W 3

X. 1. The bisectors of the angles B and C of $\triangle ABC$ meet in I. If $\angle BIC = 140^\circ$, prove that $\angle A = 100^\circ$.

X. 2. The side of BC of par^m ABCD is produced to X. Prove that quad^r ACXD = $\triangle$ DEX in area.

X. 3. Draw a par^m ABCD having $AB = 2.7''$, $BC = 3.8''$, $\angle B = 65^\circ$. E is a point in AD and F is a point in CD such that $AE = 2.4''$ and $DF = 1.8''$. Construct a $\triangle BFG$ equal in area to $\triangle EFB$, having G on BA produced. Measure BG. Also, without measuring any other lines, find the ratios of the areas of $\triangle$ s BCF and EDF to the area of ABCD.

REVISION EXAMPLES.

(Including Pythagoras.)

Y. 1. In quad^l ABCD, $AB = 3''$, $BC = 4''$, $DA = 13''$, $\angle B = 90^\circ$, $\angle ACD = 90^\circ$. Calculate the area of the quadrilateral.

Y. 2. In $\triangle ABC$, $AB = 7$ cm., $AC = 9$ cm.; CY, the altitude through C, is 6 cm.; using areas, find BX, the altitude through B.

Y. 3. ABC is an acute-angled triangle, and AX is the altitude through A. If $BX = 3 XC$, show that $AB^2 - AC^2 = 8 XC^2$.

Z. 1. In the quadrilateral ABCD, $AB = 9$ in., $BC = 12$ in., $CD = 25$ in., $DA = 20$ in., $\angle B = 90^\circ$. Prove, without a drawing to scale, that $\angle DAC$ is 90° .

Z. 2. ABCD is a para^m, and O the point of intersection of AC and BD. If X is a point in OB, prove that $\triangle AXB = \triangle XBC$ in area.

Z. 3. ABC is an equilateral triangle, and AX an altitude. If XC is produced to Y so that $CY = BC$, prove that $AY^2 = 3 AC^2$. (Take BC as 2x units.)

A.A. 1. In the trapezium PQRS, $PQ \parallel SR$ and $PQ > SR$. ST is drawn $\parallel RQ$ to meet PQ at T; SM bisects PT at M. Prove that quad^l SPMR = $\frac{1}{2}$ of quad^l PQRS in area.

A.A. 2. On the side QR of the acute-angled triangle PQR a square QRST is drawn on the opposite side of QR from P. Prove that $PT^2 - PS^2 = PQ^2 - PR^2$. (Draw $PXY \perp QR$ and TS.)

A.A. 3. DBCE is a straight line in which $DB = CE$ and ABC is an isosceles triangle on BC as base. The angle BAC is bisected by AF and F is a point on AF between A and BC. Prove that $DF = EF$ and $\angle ABF = \angle ACF$.

B.B. 1. In the rhombus ABCD, $AC = 6$ cm. and $BD = 8$ cm. Calculate its area and the perpendicular distance between a pair of opposite sides.

B.B. 2. ABCD is a square of area 100 sq. in.; E, F, G, H are points on AB, BC, CD, DA respectively, so that EFGH is a square of area 52 sq. in. If $AE = x$ in., prove that $x(10 - x) = 24$, and deduce the possible lengths of AE and EB.

B.B. 3. In the figure to Theorem 29 prove that $CD = AK$ and that $CE^2 - AH^2 = AB^2 - BC^2$.

C.C. 1. Prove that a triangle whose sides (in ins.) are $a^2 - b^2$, $a^2 + b^2$, $2ab$ is right-angled. Also calculate its area.

C.C. 2. In a regular hexagon ABCDEF (centre O) $AB = 4$ in.; OM is drawn $\perp$ to AB, and OB, AC meet in N. Calculate (1) OM, (2) AC, (3) BN, (4) the area of the hexagon. (Surds need not be worked out.)

C.C. 3. In $\triangle ABC$ $\angle A$ is 90° ; X bisects AC, and Y bisects AB. Prove that $BX^2 + CY^2 = 5 XY^2$.

PRACTICAL EXAMPLES, INCLUDING THE USE OF LOCI.

1. Draw a triangle ABC on a base of 2 in., with the $\angle B = 55^\circ$ and having a height (drawn through A) of 1.5 in. Measure AC .

2. Construct a parallelogram whose diagonals are 7 cm. and 10 cm., the angle between them being 70° . Find its area.

3. Draw $\triangle ABC$, given $AB = 3$ in., $AC = 2.3$ in., $\angle B = 60^\circ$, and find its area.

4. Draw a triangle whose sides are 2 in., 2.5 in., 3 in. and find its area by taking each side in turn as base and measuring the corresponding altitude.

5. Construct a para^m $ABCD$ of area 8 sq. in. such that $AB = 4$ in., $AC = 3.5$ in., $\angle DAB$ being acute. Measure $\angle B$.

6. Draw a quad^l $ABCD$, with $AB = 7$ cm., $BC = 8$ cm., $CD = 5$ cm., $DA = 6$ cm., $\angle B = 60^\circ$. Now draw a $\triangle$ of equal area having one side of 7 cm. and one angle of 60° . Calculate its area.

7. In $\triangle ABC$, $BC = 2.5$ in., $CA = 2.1$ in., $\angle C = 52^\circ$. By drawing two loci find the position of a point P which is equidistant from A and B , and which is equidistant from AB and BC .

8. Draw the $\triangle ABC$ with $AB = 2.8$ in., $BC = 1.9$ in., $AC = 3.5$ in. Find a point D inside the triangle which is equidistant from AB and BC , and such that $AD = 3.2$ in.

9. Construct $\triangle ABC$, given $\angle A = 75^\circ$, $\angle C = 40^\circ$, and the length of the altitude through B 6.8 cm. Measure AC .

10. Draw a parallelogram whose non-parallel sides are 7 cm. and 5 cm. long and such that one diagonal is 10 cm. long. Measure its angles.

11. Draw $\triangle ABC$, given $AB = 2.1$ in., $BC = 2.7$ in., $\angle B = 50^\circ$. Now draw a triangle of equal area XBY , having $XB = 1.7$ in. and $\angle B = 50^\circ$. Measure BY .

12. Construct $\triangle ABC$ in which $BC = 6$ cm., the altitude through A 4 cm., and the median through A 4.5 cm.

13. Draw a figure to represent a rectangular table 12 ft. by 9 ft. and draw the complete locus of a point on its surface which is always 3 ft. away from the nearest edge of the table.

REVISION EXAMPLES.

(Including chords and angles in a circle.)

D.D. 1. A chord of a circle 10 in. away from the centre is 22 in. long. How far from the centre is a chord of the same circle which is 10 in. long?

D.D. 2. AB is a diameter of a circle and CD a parallel chord. The angle DAB is 27° ; calculate the angles DCA, DAC.

D.D. 3. AD, BE are altitudes of $\triangle ABC$. AD is produced to meet the circle through A, B, C in F, and FB is joined. Prove that $\angle EBC = \angle FBC$.

E.E. 1. Two parallel chords AB, CD of a circle are drawn, their lengths being 104 cm. and 120 cm. and the centre of the circle being between them. If the radius of the circle is 85 cm., calculate how far the chords are apart.

E.E. 2. ABC is an acute-angled triangle inscribed in a circle; BA is produced to D so that $AD = AC$ and P is any point in the arc BAC. Prove that $\angle BPC = 2 \angle BDC$.

E.E. 3. Two radii OA, OB are drawn in a circle so that $\angle AOB = 74^\circ$. BC is a chord $\parallel$ AO, and AB and OC when produced meet in D. Calculate $\angle BDC$, and find out which is the longer BD or the radius of the circle.

F.F. 1. In Fig. FF 1 with the angles shown, calculate the angles EDA, OAC, AOD.

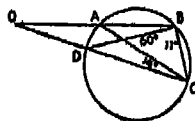


FIG. FF1

F.F. 2. In the cyclic quadrilateral ABCD the diagonal AC bisects $\angle DAB$. Prove that $\angle CDB = \angle CBD$.

F.F. 3. $\triangle ABC$ is isosceles having $AB = AC$ and $\angle B = 2 \angle A$. Find the number of degrees in angles A and B. If the bisector of $\angle C$ cuts AB in D, prove that $AD = DC = BC$.

G.G. 1. ABCD is a cyclic quadrilateral, having $AD \parallel BC$. The point E lies on the arc BC on the opposite side of BC from D, and DE meets BC at F. Show that $\angle FBE + \angle DFC = \angle ADC$.

G.G. 2. AB and CD are two equal chords of a circle, centre O, and when produced they meet outside the circle in P. By drawing perpendiculars from O to the chords, prove that P is the same distance from the mid-points of the chords.

G.G. 3. ABCD and AXCY are par^{ms} with a common diagonal AC; prove that AC, BD and XY meet in a point.

H.H. 1. In the cyclic quadrilateral ABCD, CA bisects $\angle BAD$. CE is drawn $\parallel$ BA to meet the circle in E and AD, or AD produced, in F. Prove that $EF = DF$.

H.H. 2. P is the mid-point of AB in $\triangle ABC$ and Q is a point on AC. CP is drawn $\parallel$ QP to meet AB, or AB produced, in R. Prove that the area of $\triangle AQR$ is one-half of the area of $\triangle ABC$.

H.H. 3. Fig. HH 3 shows equal arcs of equal circles on a common base AB 10 in. long. The greatest width CD of the figure is 4 in. Calculate the radius of each circle.

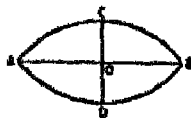


FIG HH 3

(Including equal arcs, tangents, contact of circles.)

I.I. 1. The length of a tangent drawn to a circle is 28 in. when it is drawn from a point A which is 35 in. from the centre O. Calculate the radius of the circle and the length of a tangent drawn from a point B which is 75 in. from O.

I.I. 2. AB and CD are equal arcs of a circle ABCD. Prove that the chords AC, BD are equal.

I.I. 3. In the quad^r ABCD, $AB = AD =$ diagonal AC; prove that
 $2 \angle BCD + \angle BAD = 360^\circ$.

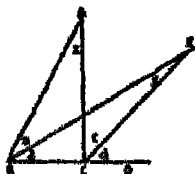


FIG I.I.

J.J. 1. In Fig. JJ 1 $a = b$ and $c = d$; by expressing x and y in terms of a and c , prove that $x = 2y$. Prove also that the centre of the circle BXC lies on the circle BAC.

J.J. 2. Two circles, of radii 2 in. and 3 in., touch externally, and both touch internally a circle of radius 7 in. Find the perimeter of the triangle formed by the centres of the circles.

J.J. 3. PA, PB are the tangents at the points A, B of a circle, centre O. AO, BO are joined and produced to meet PB, PA, both produced, in C and D respectively. Prove that A, B, C, D lie on a circle.

K.K. 1. In $\triangle ABC$, $\angle A = 68^\circ$ and $\angle B = 34^\circ$. The circle ABC is drawn and L is the mid-point of the minor arc BC; LM is a diameter of the circle. Find the $\angle$ s ACL, ACM.

K.K. 2. ABCD is a square inscribed in a circle, and X is a point on the minor arc BC. Prove that $\angle BXC = 3 \angle AXB$.

K.K. 3. AB is a chord of a circle (centre O), and BT is the tangent at B; OT, drawn $\perp$ OA, meets BT in T and AB in C. Prove that $TE = TC$.

L.L. 1. A, B, C, D are on the circumference of a circle, centre O; $AB = AO = AC$ and $BC = BD$. Prove that $\triangle BCD$ is equilateral.

L.L. 2. In the cyclic quadrilateral ABCD, $BC \parallel AD$. The point F lies on the circumference of the circle ABCD between B and C on the opposite side of BC from D, and DF meets BC at E. Prove that $\angle FBE + \angle DEC = \angle ADO$.

L.L. 3. A semicircle is drawn on AOB as diameter, O being the centre and the radius being R in. Semicircles are now drawn inside the other on AO and OB as diameters. A circle is now drawn to touch the large circle internally and the small circles externally. Prove that its radius must be $\frac{1}{2}R$ in.

(Including the alternate segment.)

M.M. 1. In $\triangle ABC$, $\angle A = 74^\circ$, $\angle B = 40^\circ$, and the incircle of the triangle touches BC in D , CA in E , and AB in F . Calculate the angle EDF .

M.M. 2. AB is a chord of a circle and AC is the tangent at A . Prove that the bisector of $\angle BAC$ bisects one of the arcs AB .

M.M. 3. Calculate the length of a common tangent drawn to two circles whose radii are 4 ft. and 3 ft. 1 in., their centres being 5 ft. 1 in. apart.

N.N. 1. Complete the following locus statements:—

The locus of the centre of a circle which

(i) passes through a point A and has a radius 2 in.
is . . .

(ii) touches both arms of the angle AOB is . . .

(iii) touches a circle of radius 4 cm. externally and has a radius of 1 cm.

N.N. 2. TA is the tangent at a point T on a circle, and ABC is a secant drawn from the point A . If the bisector of the angle BTC meets BC in D , prove that $\angle ATD = \angle ADT$.

N.N. 3. Two intersecting circles have radii 5 in. and $3\frac{1}{2}$ in., and the length of the common chord is 8 in. Find the distance between the centres if the common chord lies between them.

O.O. 1. A circle meets a second circle in A and B , and the second meets a third circle in C and D . The line AC , when produced, meets the first circle in E and the third in F ; the line BD , when produced, meets the first circle in G and the third in H . Prove that the quadrilateral $EFHG$ is cyclic.

O.O. 2. The circles ABP , ABQ meet in A and B , and are such that AP and AQ , the tangents at A to the circles ABQ , ABP respectively, are at right angles. Prove that PBQ is a straight line.

O.O. 3. What (in each case) is the locus of the centre of a circle which

(i) touches a given line AB at the point A ?

(ii) touches a given line AB and has a given radius of r in.?

(iii) passes through two given points X and Y ?

P.P. 1. ABC is a straight line and T is a point outside ABC. Complete the statement: "CT is the tangent at T to the circle ATB if the $\angle BTC$ is equal to . . ."

If $\triangle ABC$, $\angle ACB = 90^\circ$, and $CD \perp AB$; E is a point on CD and AE is produced to meet the circle on AB as diameter in F. Prove that AC is the tangent at C to the circle CEF.

P.P. 2. In $\triangle ABC$, $AB = 7$ cm., $BC = 8$ cm., $CA = 9$ cm. Three circles are drawn, to touch each other externally in pairs, their centres being A, B, C and their radii (in cm.) being x , y , z respectively. Write down three equations, each connecting two of the letters x , y , z , and by solving them, find the radii of the circles.

P.P. 3. What is the locus of

- (i) the vertex A of $\triangle ABC$ drawn on a given base BC and having a known vertical angle at A of 35° ?
- (ii) a point P, the tangent from which to a given circle has a definite length, say 1 in.?
- (iii) the mid-point of a chord 2 in. long of a given circle?

Q.Q. 1. A circle is drawn through the corners A, C, D of a parallelogram ABCD. The tangent to this circle at C meets AB, or AB produced, at E. Prove that $\angle ACS = \angle ABC$.

Q.Q. 2. ABCD is a cyclic quadrilateral whose diagonals AC, BD cut at right angles at E (not the centre). The tangents to circle ABCD at A and D meet at P, and the tangents at B and C meet at Q. Prove that $\angle P + \angle Q = 180^\circ$.

Q.Q. 3. The tangents from a point A to a circle are AB and AC. The tangent at a third point D meets AB in X and AC in Y. If $XD = DY$, prove that AD bisects $\angle BAC$.

THE CIRCLE—REVISION OF PRACTICAL CONSTRUCTIONS.

1. If an arc of an incomplete circle is given you, how would you find its centre?

2. Construct two circles of radii 1 in., 1.5 in., such that their common chord is 1.8 in. long.

3. Draw a triangle whose sides are 1 in., 2.4 in., 2.6 in., and draw a circle through its angular points. Where is the centre of the circle?

4. Draw a circle of radius 5 cm. and take a point P 7 cm. from the centre. Draw one of the tangents from P to the circle and measure its length.

5. Draw a circle of radius 1.8 in. and take a point P on it. Draw the tangent at P, and then a second tangent inclined to the former at 84° .

6. Draw a triangle whose sides are 1·8 in., 2·2 in., 3·2 in. and draw a circle to touch the sides internally. Measure its radius.

7. In the triangle given in No. 6 draw a circle to touch the shortest side externally and the other two sides produced.

8. $\angle ABC$ is an angle of 70° ; draw it, and draw a circle with radius $\frac{1}{2}$ in. to touch AB and BC .

9. Draw a circle, centre O and radius 5 cm.; now draw any radius OA , and produce it to B so that $OB = 6\cdot5$ cm. Construct a circle of radius 3 cm. to go through B and to touch the given circle externally.

10. Draw $\triangle ABC$ with $BC = 7$ cm., $\angle ABC = 48^\circ$, and $\angle ACB = 61^\circ$. Now draw a circle which shall pass through A and touch BC at C .

11. AB and CD are parallel lines met by a transversal AC 1·4 in. long and such that $\angle BAC = 70^\circ$. Construct a circle to touch AB , AC , and CD .

12. Draw a semicircle on a diameter AB which is 3 in. long. Now draw a circle inside the semicircle to touch the curve and AB , and to have a radius of $\cdot 6$ in.

13. AB is a given line 1·2 in. long. On it as base draw a triangle ABC having $\angle C = 65^\circ$ and with an area of $\cdot 6$ sq. in.

14. Draw circles of radii 1·5 in. and 0·9 in. touching externally; now draw a third circle of radius $\cdot 7$ in. to touch each of the other circles externally.

15. Draw a triangle ABC in which the base $BC = 2\cdot2$ in., the angle A is 70° , and the median bisecting BC is 1·3 in.

16. Draw a triangle ABC , given $BC = 7$ cm., $\angle A = 55^\circ$, and $AC = 8$ cm. How many different triangles are possible?

17. A semicircle is drawn on the line AB , 2 in. long, as diameter, and AB is produced both ways to C and D . Find a construction for drawing a circle of radius $\cdot 4$ in. which shall touch the line CD and also touch the semicircle, internally or externally. How many circles are possible?

18. Draw two parallel straight lines AB , CD with a line AC perpendicular to both, AC being 2 in. long. E is a point in AC such that $AE = \cdot 7$ in. Now draw a circle which passes through E and which touches the two parallels.

REVISION EXAMPLES.

(Including extensions of Pythagoras, Apollonius, intersecting chords.)

R.R. 1. Discuss whether the following triangles are right-angled, obtuse-angled, or acute-angled, the lengths of the sides being given in each case :—

(i) 11, 12, 17; (ii) 13, 14, 19; (iii) 11, 60, 61

R.R. 2. In each of the Δ s ABC, AX is an altitude. Calculate BX.

(i) AB = 7 in., AC = 4 in., BC = 10 in.

(ii) AB = 5 cm., BC = 6 cm., CA = 9 cm.

R.R. 3. ABC is an equilateral triangle of side 6a, and D is a point in BC such that $BD = \frac{1}{3} DC$. Prove by Pythagoras that $AD^2 = 7 BD^2$.

S.S. 1. Calculate the area of a triangle whose sides are AB = 13 in., BC = 14 in., and CA = 15 in. as follows :—

Calculate the projection of AB on BC and then the altitude through A.

S.S. 2. (i) If the lengths of the sides of a triangle are (in cm.) 5.5, 6.5, 8, find the length of the median which bisects the longest side.

(ii) If the sides of a triangle are $2a$, $3a$, $2x$, and the median to the third side is $2a$, prove that $2x^2 = 5a^2$.

S.S. 3. ABC and ADE are straight lines such that

AD = BD = BE = CE. Prove that $\angle CEF = 4 \angle BAD$.

T.T. 1. In ΔABC , AD is a median and E bisects AD. Prove that $2(BE^2 + CE^2) = AD^2 + BC^2$.

T.T. 2. AB is a diameter of a circle (centre O). D is taken on the circle so that chord DB = the radius. The tangent at D meets AB produced at C. Prove that AD = DC.

T.T. 3. From an external point B, BA is drawn as a tangent, and BCD a secant, to a circle. If AB = 8 in. and BC = 4 in., calculate CD.

U.U. 1. A and B are points 12 cm. apart and P is a point which moves so that $PA^2 + PB^2 = 170$ sq. cm. O is the mid-point of AB. Calculate OP, and state clearly how the point P moves.

U.U. 2. X is a point 11 in. from the centre O of a circle of radius 14 cm.; AXB is a chord such that $AX = 3XB$. Find the length of AB.

U.U. 3. ΔABC is inscribed in a circle. If AB = AC = 10 in., and BC = 16 in., calculate the radius of the circle. (Join the centre to A and B.)

REVISION EXAMPLES.

(Including ratios of parallel lines, bisectors of angles of triangle, similar triangles.)

V.V. 1. (i) A line AB, 15 in. long, is divided, internally at X and externally at Y, in the ratio 4:1. Find the length of XY.

(ii) In $\triangle ABC$, $AB = 2''$, $BC = 3''$, $CA = 4''$. The internal and external bisectors of $\angle B$ meet AC, or AC produced, in P and Q. Calculate AP and AQ.

V.V. 2. In Fig. VV 2 $\angle AXC = \angle BYA = \angle CZB$. Prove that $\angle E$ of $\triangle DEF = \angle ABC$, and show that $\triangle DEF, ABC$ are similar.

V.V. 3. In $\triangle ABC$, $BC = 9\frac{1}{2}$ in.; PQ and RS are drawn $\parallel BC$, meeting AB in P and R, and AC in Q and S. If $AP = SC = 3$ in., $RB = QS = 2$ in., calculate PR, AQ and RS.

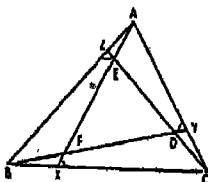


FIG. VV 2

W.W. 1. In $\text{par}^m ABCD$, $AB = 3$ in., $BC = 5$ in. The bisector of $\angle BAD$ meets BD at E, and the bisector of $\angle ABC$ meets AC at E. Find (in turn) the ratios of each of the $\triangle AFB, AEB$ to the area of the par^m . What can you now deduce about EF?

W.W. 2. The side CD of a square ABCD is 5 cm. long and in CD a point E is taken so that $CE = 3$ cm. BE and AD are produced to meet in F. Calculate DF and area of $\triangle DBF$.

W.W. 3. The bisector of $\angle BAC$ meets BC at D; X is taken on AD, or AD produced, so that $CX = CD$. Prove by similar triangles that $AB : BD = AC : XC$.

X.X. 1. The $\text{par}^m ABCD$ has an area of 12 sq. in. P and Q are taken on AB so that $AP = PQ = QB$, and DP, CQ are produced to meet in X. Calculate

(i) the area PDC; (ii) the area PQC; (iii) the ratio XP/PD ; (iv) the area XPQ.

X.X. 2. The bisector of $\angle A$ of $\triangle ABC$ meets BC in D, and AE, draw $\perp AD$, meets BC produced in E. Prove that $BD : DC = BE : EC$.

X.X. 3. A point E is taken on the diagonal BD of a cyclic quad^l ABCD so that $\angle BAE = \angle CAD$. Prove that $BA : BE = CA : CD$.

Y.Y. 1. In the $\text{par}^m ABCD$, E is a point on AB and F a point on BD such that $EF \parallel AD$; also a line through F drawn $\parallel DC$ meets EC at G. Prove that $EG \parallel AC$.

Y.Y. 2. Two circles CAB , DAB have a line CAD drawn through A . By joining C and B to the centre of circle CAB , and D and B to the centre of circle DAB , prove that

$CB : BD$ in the ratio of the radii.

Y.Y. 3. The bisector of $\angle A$ of para^m $ABCD$ meets BD at P and CD at Q ; prove that $AP : PQ = AB : AD$.

(Including intersecting chords, tangent and secant, areas of similar triangles.)

Z.Z. 1. $ABCD$ is a trapezium with $AB \parallel DC$ and AC, BD meet in Q . If $AB : DC = 3 : 2$, find the ratios DQ/OB ; $\Delta AOD/\Delta COD$; $\Delta AOB/\Delta COD$.

Z.Z. 2. AB and CD are two chords of a circle which meet, when produced, in X . Show that $\angle XAC : \angle XBD = \angle XAD : \angle XBC$.
 $XA^2 : XD^2 = XA \cdot XC : XB \cdot XD$.

Z.Z. 3. The diagonals of a cyclic quad^l $ABCD$ cut at O . Prove that $AD \cdot OC = BC \cdot OD$.

α . 1. In Fig. α 1 AB is a tangent. Calculate x and y .

α . 2. The base BC of an equilateral triangle ABC is produced to X (beyond B) and Y (beyond C) so that $\angle XAY = 120^\circ$. Prove that

$$XB \cdot XY = XA^2.$$

α . 3. The internal bisectors of the three angles of ΔABC meet in I , and HIK is drawn $\perp AI$ to meet AB in H and AC in K . Prove that $\angle AHK = \frac{1}{2}(B + C)$, and that $\Delta s BHI, CKI$ are similar.

β . 1. From T , a point outside a circle, a tangent TA and secant TBC are drawn. By considering the areas of similar triangles having AB and AC as corresponding sides, prove that

$$AB^2 : AC^2 = TB : TC.$$

β . 2. $ABCD$ is a cyclic quad^l, and DA, CB meet, when produced, in O . OP is drawn $\parallel AB$ to meet BC in P . Prove that

$$OD^2 = OP \cdot OC.$$

β . 3. AB is a flagstaff fixed on the top of a vertical pole BC , and at a point O on the ground (level with C) the flagstaff and pole subtend equal angles. If $AB = 13$ ft. and $BC = 12$ ft., calculate OC, OA .

γ . 1. The chord AB of a circle is parallel to the diameter CD . The tangent at D meets AB produced in E , and the bisector of $\angle CDE$ meets AB in F . Prove that $AF : FB = AD : BD$.

γ . 2. Points P, Q, R are taken on the sides BC, CA, AB respectively of ΔABC , so that $BP = 2 PC, CQ = 3 QA, AR = 4 RB$. Find the ratios $\Delta BPR : \Delta ABC$ and $\Delta PQR : \Delta ABC$.

γ . 3. In ΔABC , I is the in-centre and J is the ex-centre corresponding to BC . Prove that

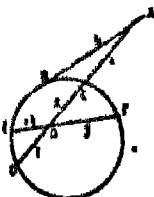


FIG. 1

